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ANALYTIC GEOMETRY

BY

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PREFACE

The author of this text believes that the differential calculus should be given to the student in college at the earliest possible moment, and that to accomplish this a short course in analytic geometry is essential. He has, therefore, written this text to supply a course that will equip the student for work in calculus and engineering without burdening him with a mass of detail useful only to the student of mathematics for its own sake. The exercises are so numerous and varied that the teacher who desires to spend a longer time on analytic geometry can easily do so; and, indeed, if more than the briefest course is given, the best way to spend the time is in working a large number of varied examples based upon the few fundamental principles which occur constantly in practice.

The author is indebted to Professor H. W. Tyler and Professor E. B. Wilson for many suggestions and to Dr. Joseph Lipka for valuable assistance both in the preparation of the manuscript and the revision of the proof.

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CONTENTS

CHAPTER 1

ALGEBRAIC PRINCIPLES

CHAPTER 2

RECTANGULAR COÖRDINATES

CHAPTER 3

STRAIGHT LINE AND CIRCLE

CHAPTER 4

SECOND DEGREE EQUATIONS

CHAPTER 5

GRAPHS AND EMPIRICAL EQUATIONS

CHAPTER 6

POLAR COÖRDINATES

CHAPTER 7

PARAMETRIC REPRESENTATION

CHAPTER 8

TRANSFORMATION OF COÖRDINATES

CHAPTER 9

COÖRDINATES OF A POINT IN SPACE

CHAPTER 10

SURFACES

CHAPTER 11

LINES AND CURVES

ANALYTIC GEOMETRY

CHAPTER 1

ALGEBRAIC PRINCIPLES

Art. 1. Constants and Variables

In analytic geometry much use is made of algebra. Hence a brief review is here given of some algebraic principles and processes used in this book.

In a given investigation a quantity is *constant* if its value is the same throughout that work, and *variable* if it may have different values. It should be noted that a quantity that is constant in one problem may be variable in another. Thus, in discussing a particular circle the radius would be constant, but in a problem about a circular disk expanding under heat the radius would be variable.

A quantity whose value is to be determined is often called an *unknown*. Such a quantity may be either constant or variable. In some cases it is not even known in advance whether it is constant or variable.

Real Numbers. — The simplest constants are numbers. The process of counting gives whole numbers. Division and subtraction give fractions and negative numbers. Whole numbers and fractions, whether positive or negative, are called *rational* numbers. A number, like $\sqrt{2}$, that can be expressed to any required degree of accuracy, but not exactly, by a fraction, is called *irrational*. Rational and irrational numbers, whether positive or negative, are called *real*.

The *absolute value* of a real number is the number without its algebraic sign. The absolute value of x is sometimes written $|x|$. Thus, $|-2| = |+2| = 2$.

Graphical Representation. — Real numbers can be represented graphically by the points of a straight line. Upon any point O of a line mark the number 0. Choose a unit of length. On one side of O mark positive numbers, on the other negative numbers, making the number at each point equal in absolute value to the distance from O to the point. The result is a *scale* on the line. When the line is horizontal, as in Fig. 1, it is usual, but not necessary, to lay off the positive numbers on the right of O, the negative numbers on the left.

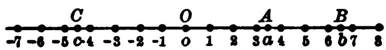

FIG. 1.

The point A representing the number a divides the scale into two parts. On one side of A, called the positive, lie all numbers greater than a; on the other side, called the negative, lie all numbers less than a. At a point B on the positive side of A is located a number b greater than a, at a point C on the negative side of A is a number c less than a.

The distance between two points of the scale is equal to the difference of the numbers at those points. This is obvious if the numbers are both positive. Thus

$$AB = OB - OA = b - a.$$

It is still true if one or both are negative. Thus, since c is negative, $CO = -c$ and

$$CB = CO + OB = -c + b = b - c.$$

Imaginary Quantities. — The extraction of roots sometimes leads to expressions like $\sqrt{-1}$ or $a + b\sqrt{-1}$, where a and b are real numbers. These expressions are called *imaginary*. This means merely that such expressions are not real numbers. It should not be inferred that imaginaries cannot be used or that they have no meaning. A quantity may have a meaning in one problem and not in another. For example, in determining the number of workmen needed in a certain undertaking the answer $3\frac{2}{3}$ would be absurd since $3\frac{2}{3}$ work-n cannot exist. In determining the area of a field the answer

-10 acres would be meaningless since there is no negative area. In determining the ratio of two lengths the answer $\sqrt{-2}$ is imaginary since the result must be a real number. But in still other problems, notably in work with alternating currents, an interpretation can be given to the process of extracting the square root of a negative number and then such results are entirely real.

Art. 2. Equations

An equation is the expression of equality between two quantities. An *identical* equation is one in which the equality is true for all values of the variables. Thus, in

$$(x-y)^2 + 4\,xy = (x+y)^2$$

the two sides are equal whatever values be assigned to x and y.

In many equations, however, the equality is true only for certain values of the variables; thus $x^2 + x = 2$ is an equation not true for all values of x, but only when $x = +1$ or -2.

Two or more equations are called simultaneous if all are satisfied at the same time. Equations often occur that are not simultaneous. Thus if $x^2 = 1$, then $x = 1$, or $x = -1$, but not both simultaneously.

A solution of an equation is a set of values of the variables satisfying the equation. Thus $x = 3$, $y = 4$ is one solution of the equation $x^2 + y^2 = 25$. A solution of a set of simultaneous equations is a set of values of the variables satisfying all of the equations.

Equivalent Equations. — Sets of equations are called *equivalent* if they have the same solutions. Thus the pair of simultaneous equations

$$x^2 + xy + y^2 = 4, \qquad x^2 - xy + y^2 = 2$$

is equivalent to

$$x^2 + y^2 = 3, \qquad xy = 1$$

(obtained by adding and subtracting the original equations) in the sense that any values of x and y, satisfying both equations of one pair, satisfy both equations of the other pair. Similarly, $(x+y)\,(x-2\,y) = 0$ is equivalent to the two equations

$$x + y = 0, \qquad x - 2\,y = 0$$

in the sense that if x and y satisfy the equation $(x + y)(x - 2y) = 0$, then either $x + y = 0$ or $x - 2y = 0$; and, conversely, if x and y satisfy either of the latter equations, they satisfy the former.

The main problem in handling equations is to replace an equation or set of equations by a simpler or more convenient equivalent set. To solve an equation or set of equations is merely to find a particular equivalent set of equations.

Degree of Equation. — The equations of algebra usually have the form of polynomials equated to zero. By a polynomial is meant an expression, such as $x^3 + x^2 - 2$ or $x^2y + 3xy - y^2$, containing only positive integral powers and products of the variables.

The degree of a term like x^3 or $3xy$ is the sum of the exponents of the variables in that term. Thus the degree of x^3 is three, that of $3xy$ is two. The degree of a polynomial is that of the highest term in it. Thus, the polynomials given above are both of the third degree.

If a polynomial is equated to zero, or if two polynomials are equated to each other, the degree of the resulting equation is that of the highest term in it. For example, $x^2 + y^2 - x = 0$ and $xy = 1$ are both equations of the second degree.

Exercises

1. Determine which of the following equations are identities:

$$(a)\ x^m x^n = x^{m+n}, \quad (b)\ \frac{x}{y} + \frac{y}{x} = 2, \quad (c)\ \frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy}.$$

2. Expand $(x + y)^6$ by the binomial theorem. Is the resulting equation an identity?

3. Show that $x = \sqrt{2}$ is a solution of the equation

$$x^5 + 2x^3 - x^2 - 8x + 2 = 0.$$

4. Show that $x = -1$, $y = 2$ is a solution of the simultaneous equations

$$x^4 + 6xy + y^4 = 5, \quad x^2 + y^2 = 5.$$

5. Show that the pair of simultaneous equations

$$x^3 + y^3 = 2, \quad x + y = 1$$

is equivalent to the pair

$$x^2 - xy + y^2 = 2, \quad x + y = 1.$$

6. Find a set of three equations equivalent to

$$(x^2 - 1)(x^3 + 2) = 0.$$

Explain in what sense the three are equivalent to the one.

7. Is $x^2 - 4xy + 3y^2 = 0$ equivalent to the pair of simultaneous equations $x = y$, $x = 3y$?

8. The symbol $\sqrt{2}$ is generally used to represent the positive square root of 2. Is $x = \sqrt{2}$ equivalent to $x^2 = 2$?

9. Show that $\sqrt{x+1} + \sqrt{x-2} = 3$ is equivalent to $x = 3$.

10. The solution of the simultaneous equations

$$x + y = 3, \quad xy = 1$$

can be written

$$x = \tfrac{1}{2}(3 \pm \sqrt{5}), \quad y = \tfrac{1}{2}(3 \mp \sqrt{5}).$$

What do these mean? How many solutions are there?

11. What is the degree of the equation $(x + y)^4 = 3xy$?

12. If x and y are the variables, what is the degree of $ax^3 = bxy$?

Art. 3. Equations in One Variable

Quadratic Equations. — The quadratic equation

$$ax^2 + bx + c = 0$$

can be solved by completing the square. Transposing c, dividing by a and adding $b^2/4a^2$ to both sides, the equation becomes

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}.$$

Extracting the square root and solving for x,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If the expression under the radical is positive, the square root can be extracted and real values are obtained for x. If it is negative, no real square root exists and the values of x are imaginary.

Solution by Factoring. — Another method for solving quadratic equations is factoring. Thus

$$x^2 + 5x - 6 = 0$$

is equivalent to

$$(x - 1)(x + 6) = 0.$$

Since a product can only be zero when one of its factors is zero, the above equation is satisfied only when $x = 1$ or $x = -6$.

If the quadratic cannot be factored by inspection, it can still be factored by completing the square. Thus

$$\begin{aligned} 3x^2 - 2x + 1 &= 3\left(x^2 - \tfrac{2}{3}x + \tfrac{1}{3}\right) = 3\left[\left(x - \tfrac{1}{3}\right)^2 + \tfrac{2}{9}\right] \\ &= 3\left(x - \tfrac{1}{3} - \tfrac{1}{3}\sqrt{-2}\right)\left(x - \tfrac{1}{3} + \tfrac{1}{3}\sqrt{-2}\right). \end{aligned}$$

The solutions of the equation $3x^2 - 2x + 1 = 0$ are then

$$x = \tfrac{1}{3}(1 \pm \sqrt{-2}).$$

In this way any equation can be solved if the expression equated to zero can be factored. For example, to solve the equation

$$x^3 + x^2 - 2 = 0$$

write it in the form

$$x^3 - 1 + x^2 - 1 = 0.$$

Since $x^3 - 1$ and $x^2 - 1$ both have $x - 1$ as a factor, the equation is equivalent to

$$(x - 1)(x^2 + 2x + 2) = 0.$$

The solutions are consequently

$$x = 1 \text{ and } x = -1 \pm \sqrt{-1}.$$

Exercises

Solve the following equations:

1. $2x^2 + 3x - 2 = 0.$
2. $x^2 + 4x - 5 = 0.$
3. $3x^2 + 5x + 1 = 0.$
4. $x^2 + x + 1 = 0.$
5. $(x^2 - 1)(x^2 - 2) = 0.$
6. $\dfrac{(x^2 - 1)}{(x^2 - 4)} = 0.$
7. $\dfrac{1}{x} + \dfrac{1}{x - 1} + \dfrac{1}{x + 1} = 0.$

Solve by factoring

8. $x^2 - 3x - 1 = 0.$
9. $2x^3 + x - 2 = 0.$
10. $x^2 - x + 1 = 0.$
11. $x^3 - 2x^2 - x + 2 = 0.$
12. $x^3 - 1 = 0.$
13. $x^4 = 1.$

14. Solve the equation $x^4 + 1 = 0$ by reducing it to the form $(x^2 + 1)^2 - 2x^2 = 0.$

15. Solve the equation $x^4 + x^2 + 4 = 0$ by the method of the last example.

16. Factor $4x^2 + 4xy - y^2$ by completing the square of the first two terms.

Art. 4. Factors and Roots

It has been shown above that the roots of an equation can be found if the factors of the polynomial equated to zero are known. Conversely, if the roots are known the factors can be found. This is done by the use of the following theorem: *If r is a root of a polynomial equation in one variable x, then $x - r$ is a factor of the polynomial.* To prove this, let

$$P = ax^n + bx^{n-1} + \cdots + px + q$$

be a polynomial of the nth degree in which $a, b, \ldots, p, q$ are constants. If r is a root of the equation given by equating this polynomial to zero,

$$ar^n + br^{n-1} + \cdots + pr + q = 0.$$

Since subtracting zero from a quantity does not change its value,

$$\begin{aligned} P &= ax^n + bx^{n-1} + \cdots + px + q - (ar^n + br^{n-1} + \cdots + pr + q) \\ &= a(x^n - r^n) + b(x^{n-1} - r^{n-1}) + \cdots + p(x - r). \end{aligned}$$

Each term on the right side of this equation is divisible by $x - r$. Hence the polynomial, P, has $x - r$ as a factor, which was to be proved.

Number of Roots. — It can be shown that any polynomial equation in one unknown has a root, real or imaginary. Assuming this, it follows that any polynomial of the nth degree in one variable is the product of n first degree factors. In fact, if r_1 is a root of $P = 0$, then

$$P = (x - r_1)\,Q,$$

Q being the quotient obtained by dividing P by $x - r_1$. Similarly, if r_2 is a root of $Q = 0$,

$$Q = (x - r_2)\,R.$$

Hence

$$P = (x - r_1)(x - r_2)\,R.$$

In the same way R can be factored, etc. Now each time a factor $x - r$ is divided out the degree of the quotient is one less. After taking out n factors, what is left will be of zero degree, that is, a constant. If a is the constant

$$P = a(x - r_1)(x - r_2) \ldots (x - r_n).$$

Hence P is the product of n first degree factors, $a(x - r_1)$, $(x - r_2)$, etc.

Since a product can only be zero when one of its factors is zero, it follows that the roots of $P = 0$ are $r_1, r_2, \ldots, r_n$. It is thus shown that *an equation of the nth degree has n roots*. Some of these r's may be equal and so the equation may have less than n distinct roots.

Rational Roots. — Though every polynomial equation in one unknown has a root, no very definite method can be given for finding it. If nothing in the particular equation suggests a better method, it is customary to try first to find a whole number or fraction that is a root of the equation. Such roots are found by trial. Some methods that may be useful are shown in the following examples.

Example 1. Solve the equation $4x^3 + 4x^2 - x - 1 = 0$.

Since x is a factor of all the terms in this equation except the last, -1, it follows that any integral value of x must be a divisor of -1. The only integral roots possible are then ± 1. By trial it is found that $x = -1$ satisfies the equation. Hence $x + 1$ is a factor of the polynomial. Factoring, the equation becomes $(x+1)(4x^2-1)=0$. The roots are consequently -1, and $\pm\frac{1}{2}$.

Ex. 2. Solve the equation $27x^3 + 9x^2 - 12x - 4 = 0$.

Proceeding as in the last example it is found that the equation has no integral root. Suppose a fraction p/q (reduced to its lowest terms) satisfies the equation. Substituting and multiplying by q^3,

$$27p^3 + 9p^2q - 12pq^2 - 4q^3 = 0.$$

Since all the terms but the last are divisible by p, and p and q have no common factor, -4 must be divisible by p. For the same reason 27 must be divisible by q. Any fractional root must then be equal to a divisor of 4 divided by a divisor of 27. It is found by trial that

$\frac{2}{3}$ is a root. Hence $x - \frac{2}{3}$ is a factor. Dividing and factoring the quotient, the equation is found to be

$$27\,(x - \tfrac{2}{3})\,(x + \tfrac{2}{3})\,(x + \tfrac{1}{3}) = 0.$$

The roots are consequently $\pm\frac{2}{3}$ and $-\frac{1}{3}$.

Exercises

Solve the following equations:

1. $x^3 - 2x^2 - x + 2 = 0.$
2. $3x^3 - 7x^2 - 8x + 20 = 0.$
3. $4x^3 - 8x^2 - 35x + 75 = 0.$
4. $8x^3 - 28x^2 + 30x - 9 = 0.$
5. $x^3 - 4x^2 - 2x + 5 = 0.$
6. $x^3 + 4x^2 + 4x + 3 = 0.$
7. $4x^4 + 8x^3 + 3x^2 - 2x - 1 = 0.$
8. $6x^4 - 11x^3 - 37x^2 + 36x + 36 = 0.$
9. $3x^4 - 17x^3 + 41x^2 - 53x + 30 = 0.$
10. $2x^4 - 9x^3 - 9x^2 + 57x - 20 = 0.$

Art. 5. Approximate Solution of Equations

If the equation has no whole numbers or fractions as roots, any real roots can still be found approximately. The method depends on the following theorem: *Between two values of x for which a polynomial has opposite signs must be a value for which it is zero.* To show this suppose when $x = a$ the polynomial is positive and when $x = b$ it is negative. Let x, beginning with the value a, gradually change. The value of the polynomial changes gradually. When x reaches b the polynomial is negative. There must have been an instant when it ceased to be positive and began to be negative. Now a number can only change gradually from positive to negative by going through zero. There is consequently a value of x between a and b for which the polynomial is zero.

The theorem can be illustrated by a figure. Let x be the number at the point M in a scale OX (Fig. 5), and let the perpendicular MP have a measure equal to the value of the polynomial for that value of x, it being drawn above OX when the value is positive, below when negative. As M moves along OX from A to B, the point P describes a curve. Since the curve is

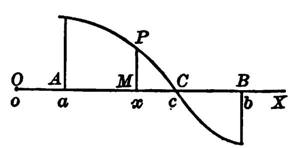

Fig. 5.

above at A and below at B, it must cross the axis at some intermediate point C. At that point the value of the polynomial is zero.

Example 1. Find the roots of $x^3 + 3x^2 - 1 = 0$ accurate to one decimal place.

By substitution the following pairs of values are found:

$$\begin{array}{rrrrrr} x = & -3, & -2, & -1, & 0, & +1, \\ x^3 + 3x^2 - 1 = & -1, & +3, & +1, & -1, & +3. \end{array}$$

The polynomial changes sign between $x = -3$ and $x = -2$, between $x = -1$ and $x = 0$ and between $x = 0$ and $x = 1$. There is consequently a root of the equation in each of these intervals. To find the root between 0 and 1, make an enlarged table for this region.

$$\begin{array}{rrrrr} x = & 0, & 0.5, & 0.6, & 1, \\ x^3 + 3x^2 - 1 = & -1, & -.125, & +.296, & 3. \end{array}$$

It is thus seen that the root is between 0.5 and 0.6. When $x = 0.55$ the polynomial is positive. Hence the root lies between 0.5 and 0.55. The value 0.5 is therefore correct to one decimal. In the same way the value -2.9 is found for the root between -2 and -3, and -0.7 for the one between -1 and 0. Since the equation can have only three roots this completes the list.

Ex. 2. Solve the equation $x^3 + x - 3 = 0$.

Since $x^3 + x$ increases with x it can equal 3 for only one real value of x. To two decimals this root is found to be 1.21. The polynomial then has $x - 1.21$ as an approximate factor. Dividing by this the quotient is

$$x^2 + 1.21\,x + 2.46.$$

The solutions obtained by equating this to zero are

$$x = -.6 \pm 1.4\sqrt{-1}.$$

Exercises

Find to one decimal the roots of the following equations:

1. $x^3 - 3x^2 + 1 = 0$.
2. $x^3 + 3x - 7 = 0$.
3. $x^3 + x^2 + x - 1 = 0$.
4. $x^4 - 3x^3 + 3 = 0$.
5. $x^4 + x - 1 = 0$.
6. $x^5 - 3x - 1 = 0$.

Art. 6. Inequalities

An inequality expresses that one quantity is greater than (>) or less than (<) another. Thus,

$$x^2+1>2x \quad \text{and} \quad (x-1)(x+2)<0$$

are inequalities. The first of these is an identical inequality (true for all values of x), the second is not. As in equations, terms can be shifted (with change of sign) from one side of an inequality to the other and inequalities having the same sign (> or <) can be added but not subtracted. Both sides of an inequality can be multiplied or divided by a positive quantity, but the sign must be changed (> to < and < to >) when an inequality is multiplied or divided by a negative quantity.

The main problem in inequalities is to determine for what values of the variable an inequality holds. How this is done is best shown by an example.

Example. Find the values of x for which

$$\frac{5x^2-x-3}{x^2(2-x)}>1.$$

This is equivalent to

$$\frac{5x^2-x-3}{x^2(2-x)}-1>0$$

or

$$\frac{(x+1)(x-1)(x+3)}{x^2(2-x)}>0.$$

The problem is to determine the values of x for which the expression on the left is positive. Since x^2 is always positive, the sign of the expression is determined by the signs of the other four factors. The values of x making one of these factors zero are $-3, -1, 1, 2$. Mark

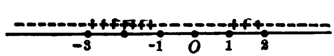

FIG. 6.

these values on a scale (Fig. 6). If $x<-3$ the three factors in the numerator are all negative, and $(2-x)$ is positive. The whole expression, having an odd number of negative factors, is negative.

If x is between -3 and -1, there are two negative factors, $x + 1$ and $x - 1$, and the whole expression is positive. If x is between -1 and $+1$, the only negative factor is $(x - 1)$ and the expression is negative. If x is between 1 and 2, all the factors are positive and the whole expression is positive. If $x > 2$ there is one negative factor, $(2 - x)$, and the expression is negative.

The expression is positive when x is between -3 and -1, or between 1 and 2. These conditions are expressed by the inequalities

$$-3 < x < -1 \quad \text{and} \quad 1 < x < 2.$$

The original inequality is equivalent to these two, in the sense that it holds when one of these does and conversely.

Exercises

Find the values of x satisfying the following inequalities:

1. $x^2 + x - 2 > 0$.
2. $x^3 > x$.
3. $x^3 - 2x^2 + 2x - 1 < 0$.
4. $x^3 - 3x - 1 < 0$.
5. $\dfrac{1}{x} + \dfrac{1}{x-2} + \dfrac{1}{x+2} > 0$.
6. Show that $x^2 - 3x + 3 > 0$ is true for all values of x.
7. Find for what values of x, the value of y is real in the equation $x^2 + xy + y^2 = 1$.
8. Find the values of x satisfying both the inequalities

$$x^2 > x, \qquad x^2 > 2.$$

Art. 7. Simultaneous Equations

Simultaneous equations in more than one unknown are solved by a process called *elimination*. This is a name applied to any process by which equations are found equivalent to the given equations but some of which contain fewer unknowns. By a continuation of this process equations may eventually be obtained each containing a single unknown and these can be solved by the methods already given. In other cases it may not be possible to solve the equations completely but they may be reduced to a simpler form. If nothing in the equations indicates a simpler way, there are three general methods that may be useful:

(1) Multiply the equations by constants or variables and add or subtract to get rid of an unknown or to obtain a simpler equation.

(2) Solve one of the equations for one of the unknowns and substitute this value in each of the other equations.

(3) Between one of the equations and each of the others eliminate the same unknown. Proceed with the new equations in the same way until finally (if possible) one of the unknowns is found. Then determine the other unknowns by substituting this value in the previous equations.

However the solutions be found they should be checked by substitution in each of the original equations.

Example 1. Solve the simultaneous equations

$$\begin{aligned} x + y + z &= 2, \\ 2x - y + 3z &= 9, \\ 3x + 2y - z &= -1. \end{aligned}$$

Adding the second to the first and twice the second to the third,

$$\begin{aligned} 3x + 4z &= 11, \\ 7x + 5z &= 17. \end{aligned}$$

Subtracting 5 times the first from 4 times the second of these equations, there is found $13x = 13$, whence $x = 1$. This value substituted in either of the preceding equations gives $z = 2$. The values of x and z substituted in either of the original equations give $y = -1$. The solution is $x = 1$, $y = -1$, $z = 2$. These values check when substituted in the original equations.

Ex. 2. Solve the equations

$$\begin{aligned} x^2 + y^2 - 2x + 4y &= 21, \\ x^2 + y^2 + x - y &= 12. \end{aligned}$$

Subtraction gives $5y - 3x = 9$. Hence $y = \frac{3}{5}(x + 3)$. This value substituted in the second equation gives

$$17x^2 + 32x - 132 = 0.$$

The roots of this are 2 and $-\frac{66}{17}$. The corresponding values of y are 3 and $-\frac{9}{17}$. The solutions are $x = 2$, $y = 3$ and $x = -\frac{66}{17}$, $y = -\frac{9}{17}$. These values check when substituted in the original equations.

Exercises

Solve the following simultaneous equations:

1. $4x - 5y + 6 = 0,$
   $7x - 9y + 11 = 0.$

2. $x + 2y - z + 3 = 0,$
   $2x - y - 5 = 0,$
   $x + 2z - 8 = 0.$

3. $x + 2y + z = 0,$
   $x - y - z = 1,$
   $2x + y - z = 0.$

4. $x + 2y + 3z = 3,$
   $x - 2y + 3z = 1,$
   $x + 4y + 9z = 6.$

5. $\frac{1}{x} + \frac{1}{y} = 1,$
   $\frac{1}{y} + \frac{1}{z} = 2,$
   $\frac{1}{z} + \frac{1}{x} = 4.$

6. $x^2 + y^2 + 2x = 0,$
   $y = 3x + 4.$

7. $h^2 + k^2 - 8h + 4k + 20 = r^2,$
   $h^2 + k^2 + 6h + 2k + 10 = r^2,$
   $h^2 + 8h + 16 = r^2.$

8. $x^2 + 4y^2 = 5,$
   $xy = -1.$

9. $x = \frac{1}{y} + \frac{1}{z},$
   $y = \frac{1}{z} + \frac{1}{x},$
   $z = \frac{1}{x} + \frac{1}{y}.$

10. $x^2 + y^2 + z^2 = 6,$
    $x + y + z = 2,$
    $2x - y + 3z = 9.$

Art. 8. Special Cases

Inconsistent Equations. — Sometimes equations are inconsistent, that is, have no simultaneous solution. This is usually shown by the equations requiring the same expression to have different values. For example, take the equations

$$\begin{aligned} x + y + z &= 1, \\ 2x + 3y + 4z &= 5, \\ x + 2y + 3z &= 3. \end{aligned}$$

Elimination of x between the first and second and first and third gives

$$y + 2z = 3, \quad y + 2z = 2.$$

Any solution of the original equations must satisfy these. Since the expression $y + 2z$ cannot equal both 2 and 3, there is no solution.

Dependent Equations. — Sometimes the solutions of part of the equations all satisfy the remaining equations. These last give no added information. Such equations are called dependent.

For example, take the equations

$$x + y = 1, \quad x^2 - y^2 + x + 3y = 2.$$

Substituting $1 - x$ for y in the second equation, it becomes $2 = 2$. All the solutions of the first equation satisfy the second. The two equations are equivalent to one equation $x + y = 1$. They have an infinite number of simultaneous solutions.

Number of Solutions. — In general, definite solutions are expected if the number of equations is equal to the number of unknowns. Thus, two equations usually determine two unknowns, three equations determine three unknowns, etc. This is, however, not always the case. The equations may be inconsistent and have no solution or may be dependent and have an infinite number of solutions. If the equations determine definite solutions, the number of solutions is expected to equal the product of the degrees of the equations. Special circumstances may, however, change this number. It can be shown that unless the number of solutions is infinite it cannot exceed the product of the degrees of the equations.

If there are fewer equations than unknowns, the unknowns will not be determined. In this case, if the equations are consistent, there will be an infinite number of solutions.

If there are more equations than unknowns, usually there will be no solution. In particular cases, however, there may be solutions. To determine whether there is a solution, solve part of the equations and substitute the values found in the remaining equations. If any of them satisfy all of the equations, there is a solution, otherwise there is none.

Homogeneous Equations. — If all the terms of an equation have the same degree the equation is called *homogeneous*. A set of homogeneous equations can often be solved for the ratios of the variables when there are not enough equations to determine the exact values.

For example, take the homogeneous equations

$$x - y - z = 0, \quad 3x - y - 2z = 0.$$

Solving for x and y

$$x = \tfrac{1}{2}z, \quad y = -\tfrac{1}{2}z.$$

Any value can be assigned to z and the values of x and y can then be determined from these equations. Let $z = 2k$. The solution is then

$$x = k, \quad y = -k, \quad z = 2k.$$

Since k is arbitrary, x, y, z have any values proportional to 1, -1, 2. This result can be written $x : y : z = 1 : -1 : 2$.

Exercises

Determine whether the following equations have no solutions, definite solutions, or an infinite number of solutions:

1. $x + y + z = 1,\quad 2x - 2y + 5z = 7,\quad 2x - 3y + 4z = 5.$
2. $x + 2y + z = 0,\quad 2x + y - z = 5,\quad 5x + 7y + 4z = 2.$
3. $x + 2y + 1 = 0,\quad 3x - y - 4 = 0,\quad 2x + 3y + 1 = 0.$
4. $x + y - z = 4,\quad x + 2y + 3z = 2,\quad 5x + 8y + 7z = 14.$
5. $x + y - 5 = 0,\quad 3x + 2y - 12 = 0,\quad 2x + y - 6 = 0.$
6. $x - y + 2z = 1,\quad 3x + y - z = 5,\quad 3x + 2y - 3z = 2.$
7. $3x + y + 1 = 0,\quad 2x + 3y - 4 = 0,\quad x + 2y - 3 = 0.$
8. $x - y = 0,\quad x^2 - y^2 = 1.$
9. $(x - y)^2 + (y - z)^2 + (z - x)^2 = 1,\quad x^2 + y^2 + z^2 = 2,\quad (x + y)^2 + (y + z)^2 + (z + x)^2 = 3.$
10. $w + x + y + z = 1,\quad w + 2x - 3y - 4z = 2,\quad w - x - 2y - 3z = 3,\quad w - 5x + 2y + 3z = 3.$

Find values to which the variables in the following equations are proportional:

11. $x + y - 2z = 0,\quad 3x - y - 4z = 0.$
12. $x + 2y + z = 0,\quad 4y + 3z = 0.$
13. $x + y - z = 0,\quad x^2 + y^2 - 5z^2 = 0.$
14. $x^2 + y^2 = 2z^2,\quad y^2 = xz.$

Art. 9. Undetermined Coefficients

It is often necessary to reduce a given expression or equation to a required form. This form is indicated by an expression or equation having letters for coefficients and the reduction is made by calculating the values of these coefficients.

In this work frequent use is made of the following theorem: *If two polynomials in one variable are equal for all values of the variable,*

the coefficients of the same power of the variable in the two polynomials are equal. To show this, suppose, for all values of x,

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n.$$

Then for all values of x

$$(a_0 - b_0) + (a_1 - b_1)x + \cdots + (a_n - b_n)x^n = 0.$$

If the coefficients in this equation are not zero, by Art. 4, it cannot have more than n distinct roots. Hence the coefficients must all be zero and $a_0 = b_0$, $a_1 = b_1$, etc., which was to be proved.

To reduce an expression to a given form, equate the expression to the given form, clear of fractions or radicals, and determine the unknown coefficients by the above theorem.

Example 1. To find the coefficients a and b such that

$$\frac{x}{(x-1)(x+3)} = \frac{a}{x-1} + \frac{b}{x+3}$$

clear of fractions, getting,

$$x = a(x+3) + b(x-1) = (a+b)x + 3a - b.$$

If this equation holds for all values of x,

$$a + b = 1, \quad 3a - b = 0.$$

Hence $a = \frac{1}{4}$, $b = \frac{3}{4}$. Conversely, if a and b have these values, the above equations are identically satisfied. Therefore

$$\frac{x}{(x-1)(x+3)} = \frac{1}{4(x-1)} + \frac{3}{4(x+3)}.$$

In many cases the expression can be more easily changed to the required form by simple algebraic processes. This is particularly the case with second degree expressions where completing the square may give the required result.

Example 2. To reduce the expression

$$1 + 4x - 2x^2$$

to the form $a - b(x-c)^2$, it can be written

$$1 - 2(x^2 - 2x) = 3 - 2(x-1)^2,$$

which is the result required.

In reducing equations to a required form it should be noted that multiplying an equation by a number gives an equivalent equation. Thus, $x + y = 1$ and $2x + 2y = 2$ are equivalent. Two equations are then equivalent when corresponding coefficients are proportional.

Example 3. To find k such that $x + 8y - 1 = 0$ and $3x - y - 3 + k(x + 3y - 1) = 0$ are equivalent, write the last equation in the form

$$(3 + k)x + (3k - 1)y - (k + 3) = 0.$$

This is equivalent to $x + 8y - 1 = 0$ if corresponding coefficients are proportional, that is, if

$$\frac{3 + k}{1} = \frac{3k - 1}{8} = \frac{k + 3}{1}.$$

These equations are satisfied by $k = -5$.

Exercises

Reduce the following expressions to the forms indicated:

1. $2x^2 + 3x + 4 = (ax + b)^2 + c.$
2. $3 + 2x - x^2 = b - (x - a)^2.$
3. $x^2 + xy + y^2 = a(x + my)^2 + b(y - mx)^2.$
4. $\dfrac{x + 1}{x(x - 2)} = \dfrac{a}{x} + \dfrac{b}{x - 2}.$
5. $\dfrac{1}{(x + 3)(x^2 + 1)} = \dfrac{ax + b}{x^2 + 1} + \dfrac{c}{x + 3}.$

Reduce the following equations to the forms indicated:

6. $3x - 4y = 5, \quad y = mx + b.$
7. $2x + 3y = 4, \quad \dfrac{x}{a} + \dfrac{y}{b} = 1.$
8. $3x^2 + 2y^2 - 6x + 4y = 1, \quad \dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1.$
9. $x^2 - 4y^2 - 4x + 8y = 4, \quad \dfrac{(x - h)^2}{a^2} - \dfrac{(y - k)^2}{b^2} = 1.$
10. $3x - y + 5 = 0, \quad (x + y - 1) + k(x - y + 3) = 0.$

Art. 10. Functions

It is often desirable to state that one quantity is determined by another. For this purpose the word *function* is used. A quantity

y is called a function of x if values of x determine values of y. Thus, if $y = 1 - x^2$, then y is a function of x, for a value of x determines a value of y. Similarly, the area of a circle is a function of its radius; for, the length of radius being given, the area of the circle is determined.

It is not necessary that a value of the variable determine a single value of the function. It may be that a limited number of values are determined. Thus, y is a function of x in the equation

$$x^2 - 2xy + y^2 + x = 1.$$

To each value of x correspond two definite values of y obtained by solving a quadratic equation.

If a single value of the function corresponds to each value of the variable, the function is called *single valued.* If several values of the function correspond to the same value of the variable the function is called *many valued.*

Kinds of Functions. — Any expression containing a variable is a function of that variable, for, a value of the variable being given, a value of the expression is determined. Such a function is called *explicit.* Thus $\sqrt{x^2 + 1}$ is an explicit function of x. Similarly, if $y = \sqrt{x^2 + 1}$, then y is an explicit function of x.

If x and y are connected by an equation not solved for y, then y is called an *implicit* function of x. For example, y is an implicit function of x in the equation

$$x^2 + y^2 + 2x + y = 1.$$

Also x is an implicit function of y.

Explicit and implicit do not denote properties of the function but merely of the way it is expressed. An implicit function is rendered explicit by solving. For example, the above equation is equivalent to

$$y = \tfrac{1}{2}(-1 \pm \sqrt{5 - 8x - 4x^2}).$$

A *rational* function is one representable by an algebraic expression

containing no fractional powers of variable quantities. For example,

$$\frac{x\sqrt{5}+3}{x^3+2}$$

is a rational function of x.

An *irrational* function is one represented by an algebraic expression which cannot be reduced to a rational form. For example, $\sqrt{x+1}$ is an irrational function of x.

A function is called *algebraic* if it can be expressed explicitly or implicitly by a finite number of algebraic operations (addition, subtraction, multiplication, division, raising to integral powers, and extraction of integral roots). All the functions previously mentioned are algebraic.

Functions that are not algebraic are called *transcendental.* For example, $x^{\sqrt{2}}$ and 2^x are transcendental functions of x.

The terms rational, irrational, algebraic, and transcendental denote properties of the function itself and do not depend on the way the function is expressed.

Notation.—A particular function of x is represented by the notation $f(x)$, which should be read function of x, or f of x, not f times x. For example, $f(x) = \sqrt{x^2+1}$, means that $f(x)$ is the definite function $\sqrt{x^2+1}$. Similarly, $y = f(x)$ means that y is a definite (though perhaps unknown) function of x.

The f in the symbol of a function should be considered as representing an operation to be performed on the variable. Thus, if $f(x) = \sqrt{x^2+1}$, f represents the operation of squaring the variable, adding 1, and extracting the square root of the result. If x is replaced by any other quantity, the same operation is to be performed on that quantity. For example, $f(2)$ is the result of performing the operation f on 2. With the above value of f,

$$f(2) = \sqrt{2^2+1} = \sqrt{5}.$$

Similarly,

$$f(y+1) = \sqrt{(y+1)^2+1} = \sqrt{y^2+2y+2}.$$

If it is necessary to consider several functions in the same discussion, they are distinguished by subscripts or accents or by the

use of different letters. Thus, $f_1(x)$, $f_2(x)$, $f_3(x)$, $f'(x)$, $f''(x)$, $f'''(x)$ (read f-one of x, f-two of x, f-three of x, f-prime of x, f-second of x, f-third of x), $g(x)$ represent (presumably) different functions of x.

Functions of Several Variables. — A quantity u is called a function of several variables if values of u are determined by values of those variables. For example, the volume of a cone is a function of its altitude and the radius of its base; for the volume is determined by the altitude and radius of base. This is indicated by the notation

$$v = f(h, r),$$

which should be read, v is a function of h and r, or v is f of h and r.

Similarly, the volume of a rectangular parallelopiped is a function of the lengths of its three edges. If a, b, and c are the lengths of the edges, this is expressed by the equation

$$v = f(a, b, c),$$

which should be read, v is a function of a, b, and c, or v is f of a, b, c.

Independent and Dependent Variables. — In most problems there occur a number of variable quantities connected by equations. Arbitrary values can be assigned to some of these quantities and the others are then determined. Those taking arbitrary values are called *independent* variables; those determined are called *dependent* variables. Which are taken as independent and which as dependent variables is usually a matter of convenience. The number of independent variables is, however, fixed by the equations.

Example. The radius r, altitude h, volume v, and total surface S of a cylinder are connected by the equations

$$v = \pi r^2 h, \quad S = 2\pi r^2 + 2\pi r h.$$

Any two of these four quantities can be taken as independent variables and the other two calculated in terms of them. If, for example, v and r are taken as the independent variables, h and S have the values

$$h = \frac{v}{\pi r^2}, \quad S = 2\pi r^2 + \frac{2v}{r}.$$

Exercises

1. If $f(x) = x^2 - 3x + 2$, show that $f(1) = f(2) = 0$.

2. If $f(x) = x + \frac{1}{x}$, find $f(x + 1)$. Also find $f(x) + 1$.

3. If $f(x) = \sqrt{x^2 - 1}$, find $f(2x)$. Also find $2f(x)$.

4. If $f(x) = \frac{x + 2}{2x - 3}$, find $f\left(\frac{1}{x}\right)$. Also find $\frac{1}{f(x)}$.

5. If $\psi(x) = x^4 + 2x^2 + 3$, show that $\psi(-x) = \psi(x)$.

6. If $\phi(x) = x + \frac{1}{x}$, show that $[\phi(x)]^2 = \phi(x^2) + 2$.

7. If $F(x) = \frac{1 - x}{1 + x}$, show that $F(a)F(-a) = 1$.

8. If $f_1(x) = 2^x$, $f_2(x) = x^2$, find $f_1[f_2(y)]$. Also find $f_2[f_1(y)]$.

9. If $f(x, y) = x^2 + 2xy - 5$, show that $f(1, 2) = 0$.

10. If $F(x, y) = x^2 + xy + y^2$, show that $F(x, y) = F(y, x)$.

11. If $f(x, y) = x^3 + 3x^2y + y^3$, show that $f(x, vx) = x^3 f(1, v)$.

12. If a, b, c are the sides of a right triangle how many of them can be taken as independent variables?

13. Express the radius and area of a sphere in terms of the volume taken as independent variable.

14. Given $u = x^2 + y^2$, $v = x + y$, determine x and y as functions of the independent variables u and v.

15. If x, y, z satisfy the equations

$$\begin{aligned} x + y + z &= 6, \\ x - y + 2z &= 5, \\ 2x + y - z &= 1, \end{aligned}$$

show that none of them can be independent variables.

16. The equations

$$\begin{aligned} x + y + z &= 6, \\ x - y + 2z &= 5, \\ 2x + 4y + z &= 13 \end{aligned}$$

are dependent. Show that any one of the quantities x, y, z can be taken as independent variable.

17. If u, v, x, y are connected by the equations

$$u^2 + uv - y = 0, \qquad uv + x - y = 0,$$

show that u and x cannot both be independent variables.

CHAPTER 2

RECTANGULAR COÖRDINATES

Art. 11. Definitions

Scale on a Line. — In Art. 1 it has been shown that real numbers can be attached to the points of a straight line in such a way that the distance between two points is equal to the difference (larger minus smaller) of the numbers located at those points.

The line with its associated numbers is called a scale. Proceeding along the scale in one direction (to the right in Fig. 11*a*) the

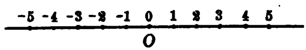

FIG. 11*a*.

numbers increase algebraically. Proceeding in the other direction the numbers decrease. The direction in which the numbers increase is called positive, that in which they decrease is called negative.

Coördinates of a Point. — In a plane take two perpendicular scales $X'X$, $Y'Y$ with their zero points coincident at O (Fig. 11*b*). It is customary to draw $X'X$, called the *x-axis*, horizontal with its positive end on the right, and $Y'Y$, called the *y-axis*, vertical with its positive end above. The point O is called the origin. The axes divide the plane into four sections called *quadrants*. These are numbered I, II, III, IV, as shown in Fig. 11*b*.

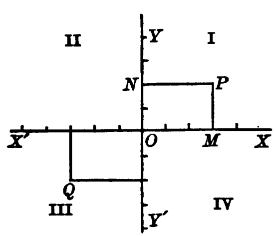

FIG. 11*b*.

From any point P in the plane drop perpendiculars PM, PN to the axes. Let the number at M in the scale $X'X$ be x and that at N in the scale $Y'Y$ be y. These num-

bers x and y are called the *rectangular coördinates* of P with respect to the axes $X'X$, $Y'Y$. The number x is called the *abscissa*, the number y is called the *ordinate* of P.

If the axes are drawn as in Fig. 11b, the abscissa x is equal in magnitude to the distance PN from P to the y-axis, is positive for points on the right and negative for points on the left of the y-axis. The ordinate y is equal in magnitude to the distance PM from P to the x-axis, is positive for points above and negative for points below the x-axis. For example, the point P in Fig. 11b has coördinates $x = 3$, $y = 2$, while Q has the coördinates $x = -3$, $y = -2$.

Notation. — The point whose coördinates are x and y is represented by the symbol (x, y). To signify that P has the coördinates x and y the notation $P\ (x, y)$ is used. For example, $(1, -2)$ is the point $x = 1$, $y = -2$. Similarly, $A\ (-2, 3)$ signifies that the abscissa of A is -2, and its ordinate 3.

Plotting. — The process of locating a point whose coördinates are given is called *plotting*. It is convenient for this purpose to use coördinate paper, that is, paper ruled with two sets of lines as in Fig. 11c. Two of these perpendicular lines are taken as axes. A certain number of divisions (one in Figs. 11c, d and e) are taken as representing a unit of length. This unit should be long enough to make the diagram of reasonable size but not so long that any points to be plotted fail to lie on the paper. By counting the rulings from the axes it is easy to locate the point having given coördinates (approximately if it does not fall at an intersection).

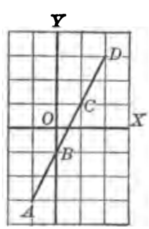

FIG. 11c.

Example 1. Show graphically that the points $A\ (-1, -3)$, $B\ (0, -1)$, $C\ (1, 1)$ and $D\ (2, 3)$ lie on a straight line.

The points are plotted in Fig. 11c. By applying a ruler to the figure it is found that a straight line can be drawn through the points.

Ex. 2. Plot the points $P_1\ (-2, -1)$, $P_2\ (3, 4)$ and find the distance between them.

The points are plotted in Fig. 11d.

Let the horizontal line through P_1 cut the vertical line through

P_2 in R. From the figure it is seen that $P_1R = 5$, $RP_2 = 5$. Consequently,

$$P_1P_2 = \sqrt{\overline{P_1R}^2 + \overline{RP_2}^2} = \sqrt{50}.$$

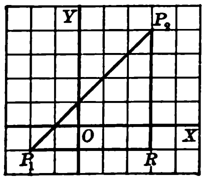

FIG. 11d.

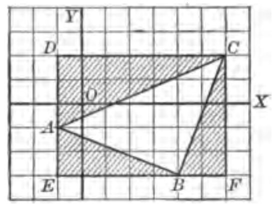

FIG. 11e.

Ex. 3. Plot the points A $(-1, -1)$, B $(4, -3)$, C $(6, 2)$ and find the area of the triangle ABC.

The points are plotted in Fig. 11e. The triangle ABC is part of a rectangle $CDEF$ with sides 5 and 7 and area $5 \times 7 = 35$. ADC, AEB and BFC are right triangles whose areas are (one-half base times altitude) $10\frac{1}{2}$, 5 and 5 respectively. The area of ABC is then

$$CDEF - ADC - AEB - BFC = 35 - 20\tfrac{1}{2} = 14\tfrac{1}{2}.$$

Exercises

1. Plot the points A $(4, 4)$, B $(-4, 4)$, C $(-4, -4)$, D $(4, -4)$ and E $(\sqrt{2}, \sqrt{3})$.

2. What are the algebraic signs of the coördinates in each of the four quadrants?

3. If a point lies on the x-axis what is its ordinate? If it lies on the y-axis what is its abscissa? Where are all the points for which $x = 1$? for which $y = -2$?

4. Let a and b be any given numbers. How is (a, b) related to each of the points $(-a, b)$, $(a, -b)$, $(-a, -b)$, (b, a), $(a + 1, b)$, $(a, b - 2)$?

5. Plot the points $(0, -2)$, $(1, 1)$, $(2, 4)$, $(-1, -5)$ and verify from the figure that they lie on a line.

6. Show graphically that the points $(5, 0)$, $(3, -4)$, $(-4, 3)$ and $(0, -5)$ lie on a circle. Find its center and radius.

7. Construct the point equidistant from $(-3, 4)$, $(5, 3)$ and $(2, 0)$. Determine its coördinates by measurement.

8. Plot the points A $(1, 2)$ and B $(3, 4)$. Let the horizontal line

through A meet the vertical line through B in P. What are the coördinates of P? What are the lengths of AP and BP? Calculate the distance AB.

9. Find the distance between A $(-2, -3)$ and B $(3, -4)$.

10. Find the area of the triangle formed by the points $(-3, 4)$, $(5, 3)$ and $(2, 0)$.

Art. 12. Segments

In this book the term line (meaning straight line) is used only when referring to the infinite line extending indefinitely in both directions. The part of a line between two points will be called a *segment*.

In many cases a segment is regarded as having a definite direction. The symbol AB is used for the segment beginning at A and ending at B. The segment beginning at B and ending at A is written BA.

The value of a segment may be any one of three things that should be carefully distinguished. Whenever the symbol AB occurs in an equation it must be understood which of these things is meant.

(1) In many cases the value AB means the length of the segment. In this case $AB = 3$ means that AB has a length of three units and $AB = CD$ means that AB and CD have equal lengths.

(2) In other cases the symbol AB represents the length together with a sign positive or negative according as the segment is directed one way or the other along the line. For example, the x-coördinate of a point is equal to the segment NP (Fig. 11*b*) considered positive when drawn to the right, negative when drawn to the left. The value of the segment is in this case a number with a positive or negative sign.

(3) Certain physical quantities, such as velocities, include in their description the direction of the lines along which they occur as well as their magnitudes and directions one way or the other along those lines. Two segments representing such quantities are equal only when they have the same length and direction. The value of such a segment (called a vector) is not a number but a number and direction.

Segments Parallel to a Coördinate Axis. — In most cases the value of a segment parallel to a coördinate axis will be the number equal

in magnitude to the length of the segment and positive or negative according as the segment is drawn in the positive or negative direc-

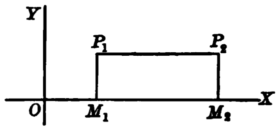

FIG. 12*a*.

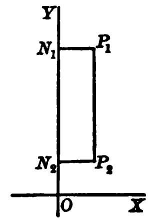

FIG. 12*b*.

tion of the axis. That is, horizontal segments are positive when drawn to the right, negative when drawn to the left; vertical segments are positive when drawn upward, negative when drawn downward.

Let P_1 (x_1, y_1) and P_2 (x_2, y_2) be the end points of a segment P_1P_2. If the segment is parallel to the x-axis (Fig. 12*a*)

$$P_1P_2 = M_1M_2 = x_2 - x_1, \tag{12}$$

for the difference $x_2 - x_1$ is equal in absolute value to the distance M_1M_2 and is positive when M_2 is on the right of M_1, negative when on the left. Similarly, if P_1P_2 is parallel to the y-axis (Fig. 12*b*), then

$$P_1P_2 = N_1N_2 = y_2 - y_1. \tag{12}$$

Therefore *in length and sign a segment parallel to a coördinate axis is represented by the difference obtained by subtracting the coördinate of the beginning from the coördinate of the end of the segment.*

Art. 13. Projection

The projection of a point A on a line MN is the foot of the perpendicular from A to MN. The projection of a segment AB is the segment $A'B'$ of MN intercepted between perpendiculars from the ends of AB. Since parallels intercept proportional distances on two lines, the lengths of two segments on a line have the same ratio as their projections. Furthermore if two segments have the same direction along a line their projections have the same direction.

For example, in Fig. 13*a* or 13*b*, in both magnitude and sign,

$$AB : BC = A'B' : B'C'.$$

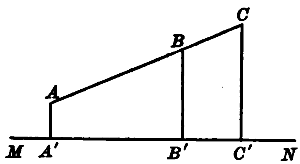

FIG. 13*a*.

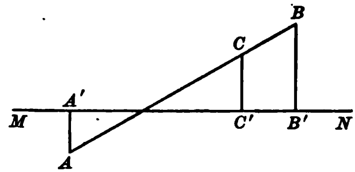

FIG. 13*b*.

In using this formula it should be noted that if AB and BC are distances, their projections must be distances and if AB and BC have algebraic signs their projections must have algebraic signs. For example, if the segments are projected on the coördinate axes and the values of the projections determined by equation (12), AB and BC must be considered opposite in sign when they have opposite directions.

Example 1. Find the middle point of the segment joining $P_1\ (x_1, y_1)$ and $P_2\ (x_2, y_2)$.

Let $P\ (x, y)$ be the middle point. Project on the axes (Fig. 13*c*). Since P is the middle point of P_1P_2, in both length and direction $P_1P = PP_2$. Hence

$$M_1M = MM_2, \quad N_1N = NN_2,$$

and so

$$x - x_1 = x_2 - x, \quad y - y_1 = y_2 - y.$$

Consequently,

$$x = \tfrac{1}{2}(x_1 + x_2), \quad y = \tfrac{1}{2}(y_1 + y_2).$$

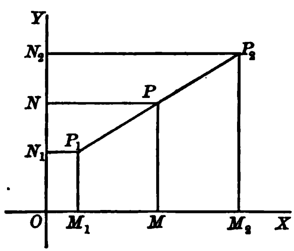

FIG. 13*c*.

The sum of several quantities divided by their number is called the average. Hence *each coördinate of the middle point is the average of the corresponding coördinates of the ends.*

Ex. 2. Given $P_1\ (-1, 1)$, $P_2\ (3, 4)$, find the point $P\ (x, y)$, on P_1P_2 produced, which is twice as far from P_1 as from P_2 (Fig. 13*d*).

Since P_1P is twice as long as PP_2 and they have opposite directions,

$$P_1P = -2\,PP_2.$$

Consequently,

$$M_1M = -2\,MM_2, \quad N_1N = -2\,NN_2,$$

and so

$$x+1 = -2\,(3-x), \quad y-1 = -2(4-y).$$

Hence $x = 7$, $y = 7$.

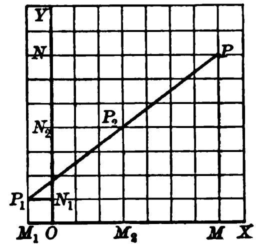

FIG. 13*d*.

Exercises

1. On the line through A $(-3, -4)$ and B $(4, 2)$ find the point two-fifths of the way from A to B.

2. The points A $(-1, -4)$, B $(0, -1)$ and C $(2, 5)$ lie on a line. Find the point P, on AC produced, such that $AB = CP$.

3. On the line through A $(1, -1)$, B $(3, 5)$ find two points each of which is twice as far from A as from B.

4. The points A $(-4, 9)$, B $(2, 0)$, C $(4, -3)$, D $(0, 3)$ lie on a line. Find the ratio of the segments AB and CD. Do these segments have the same or opposite directions?

5. On the line through A $(2, -1)$ parallel to the line through B $(1, 1)$ and C $(4, 5)$ find two points each of which is at the distance BC from A.

6. Given three points, P_1 (x_1, y_1), P_2 (x_2, y_2), P_3 (x_3, y_3), find the middle point Q of P_1P_2, then find the point R one-third of the way from Q to P_3. Show that each of its coördinates is the average of the corresponding coördinates of P_1, P_2 and P_3.

Art. 14. Distance between Two Points

Let the points be P_1 (x_1, y_1) and P_2 (x_2, y_2) (Fig. 14*a* or 14*b*). Let the projections of P_1P_2 on the axes be M_1M_2 and N_1N_2. Let the lines P_1N_1 and P_2M_2 intersect in R. In the right triangle P_1RP_2

$$P_1P_2 = \sqrt{P_1R^2 + RP_2^2}.$$

Now $P_1R = M_1M_2$ and the distance between two points of a scale is equal to the positive difference of their coördinates. Consequently, $P_1R^2 = (x_2 - x_1)^2$. Similarly, $RP_2^2 = (y_2 - y_1)^2$. Therefore

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{14}$$

In this formula, since $x_2 - x_1$ is squared it is immaterial whether it is written $x_2 - x_1$ or $x_1 - x_2$. Similarly, instead of $y_2 - y_1$ can

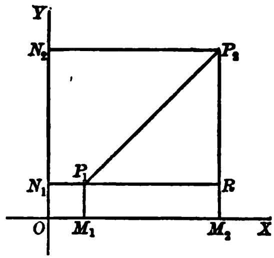

FIG. 14*a*.

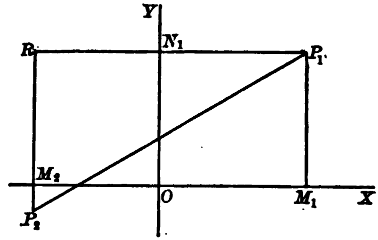

FIG. 14*b*.

be put $y_1 - y_2$. The formula expresses that *the distance between two points is equal to the square root of the sum of the squares of the differences of corresponding coördinates.*

Example 1. Find the distance between the points A $(-1, 1)$ and B $(1, -2)$.

By the formula

$$AB = \sqrt{(1+1)^2 + (-2-1)^2} = \sqrt{13}.$$

Ex. 2. Find the points 2 units distant from the x-axis and 5 units distant from the point A $(1, 2)$.

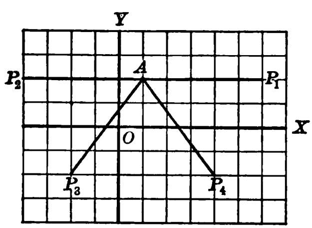

FIG. 14*c*.

Let P (x, y) be such a point (Fig. 14*c*). Since $PA = 5$

$$(x-1)^2 + (y-2)^2 = 25.$$

Since the point is two units distant from the x-axis

$$y = \pm 2.$$

Substitution of 2 for y in the previous equation gives $x = 6$ or -4. Substitution of -2 for y gives $x = 4$ or -2. There are consequently four points P_1 $(6, 2)$, P_2 $(-4, 2)$, P_3 $(-2, -2)$, P_4 $(4, -2)$ which satisfy the conditions.

Ex. 3. Find a point equidistant from the three points, A $(9, 0)$, B $(-6, 3)$ and C $(5, 6)$.

Let $P(x, y)$ be the point required (Fig. 14*d*). Since $PA = PB$

$$\sqrt{(x-9)^2 + y^2} = \sqrt{(x+6)^2 + (y-3)^2}.$$

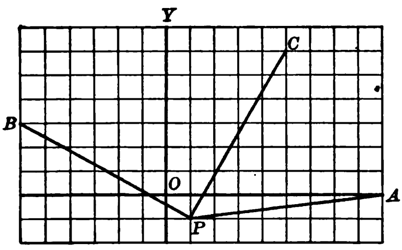

FIG. 14*d*.

Squaring and cancelling,

$$5x - y = 6.$$

Similarly, since $PA = PC$,

$$2x - 3y = 5.$$

Solving simultaneously gives $x = 1$, $y = -1$. The required point is therefore $(1, -1)$.

Exercises

1. Find the perimeter of the triangle whose vertices are (2, 3), (−3, 3) and (1, 1).

2. Show that the points (1, −2), (4, 2) and (−3, −5) are the vertices of an isosceles triangle.

3. Show that the points (0, 0), (3, 1), (1, −1) and (2, 2) are the vertices of a parallelogram.

4. Given A (2, 0), B (1, 1), C (0, 2) show that the distances AB, BC and AC satisfy the equation $AB + BC = AC$. What do you conclude about the points?

5. Show that (6, 2), (−2, −4), (5, −5), (−1, 3) are on a circle whose center is (2, −1).

6. It can be shown that four points form a quadrilateral inscribed in a circle if the product of the diagonals is equal to the sum of the products of the opposite sides. Assuming this, show that (−2, 2), (3, −3), (1, 1) and (2, 0) lie on a circle.

7. Find the coördinates of two points whose distances from (2, 3) are 4 and whose ordinates are equal to 5.

8. Find a point on the x-axis which is equidistant from (0, 4) and (−3, −3).

9. Find the center of the circle passing through (0, 0), (−3, 3) and (5, 4).

10. Given A (0, 0), B (1, 1), C (−1, 1), D (1, −2), find the point in which the perpendicular bisector of AB cuts the perpendicular bisector of CD.

11. Find the foot of the perpendicular from (1, 2) to the line joining (2, 1) and (−1, −5).

Art. 15. Vectors

A vector is a segment of definite length and direction. The point P_1 is the beginning, the point P_2 the end of the vector P_1P_2. The direction of the vector along its line is often indicated by an arrow as shown in Fig. 15*a*.

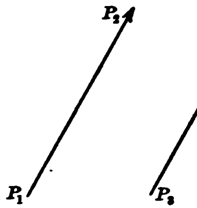

FIG. 15*a*.

Two vectors are called equal if they have the same length and direction. For example, in Fig. 15*a*, $P_1P_2 = P_3P_4$. If two vectors have the same length but opposite directions, either is called the *negative* of the other. For example, $P_1P_2 = -P_4P_3 = -P_2P_1$.

Let the projections of P_1P_2 on the coördinate axes be M_1M_2 and N_1N_2 (Fig. 15*b*). The *x-component* of the vector P_1P_2 is defined as the length of M_1M_2 or the negative of that length, according as M_1M_2 has the positive or negative direction along the x-axis. Similarly, the *y-component* is the distance N_1N_2 or the negative of that distance, according as N_1N_2 is drawn in the positive or negative direction along the y-axis. For example, in Fig. 15*b*, the components of P_1P_2 are 3 and 4, while those of P_2P_3 are -7 and 3.

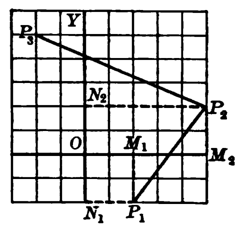

FIG. 15*b*.

Let the points be $P_1\ (x_1, y_1)$, $P_2\ (x_2, y_2)$. In Art. 12 it has been shown that in magnitude and sign M_1M_2 and N_1N_2 are represented by $x_2 - x_1$ and $y_2 - y_1$. Hence $x_2 - x_1$ is the x-component and $y_2 - y_1$ is the y-component of P_1P_2. That is, the components of a vector are obtained by subtracting the coördinates of the beginning from the corresponding coördinates of the end of the vector.

If two vectors are equal their components are equal. For let $P_1P_2 = P_3P_4$ (Fig. 15*c*). Then, since by definition of equality P_1P_2 and P_3P_4 have the same length and direction, the triangles

P_1RP_2, P_3SP_4 are equal and corresponding sides have the same direction. Consequently, in both length and sign,

$$M_1M_2 = P_1R = P_3S = M_3M_4,$$
$$N_1N_2 = RP_2 = SP_4 = N_3N_4.$$

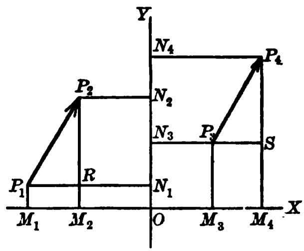

FIG. 15c.

Conversely, if the components are equal, the triangles are equal and corresponding sides have the same direction. Consequently, the vectors are equal.

Notation. — In this book the vector whose components are a and b will be represented by the symbol $[a, b]$. To signify that the vector u has the components a and b, the notation $u\,[a, b]$ will be used. For example, $P_1P_2 = [-2, 3]$ means that the vector P_1P_2 has an x-component equal to -2 and a y-component equal to 3. If the points P_1 and P_2 are (x_1, y_1) and (x_2, y_2), the components of P_1P_2 are $x_2 - x_1$ and $y_2 - y_1$ and

$$P_1P_2 = [x_2 - x_1, \quad y_2 - y_1].$$

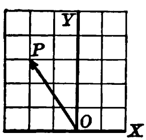

FIG. 15d.

Example 1. Construct a vector equal to $[-2, 3]$.

The vector OP from the origin to the point $P\,(-2, 3)$ is

$$OP = [-2-0, 3-0] = [-2, 3].$$

Hence OP is a vector of the kind required (Fig. 15d).

Ex. 2. Show that the points $A\,(1, 3)$, $B\,(2, 1)$, $C\,(3, 4)$, $D\,(4, 2)$ form the vertices of a parallelogram.

The vectors AB and CD are equal. For

$$AB = [2-1, 1-3] = [1, -2],$$
$$CD = [4-3, 2-4] = [1, -2].$$

That is, AB and CD are parallel and have the same length. Consequently $ABCD$ is a parallelogram (Fig. 15e).

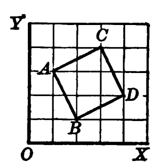

FIG. 15e.

Ex. 3. Find the area of the triangle PP_1P_2, given

$$PP_1 = [a_1, b_1], \quad PP_2 = [a_2, b_2].$$

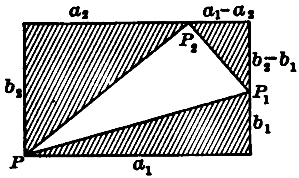

FIG. 15*f*.

In Fig. 15*f* the sides of the shaded triangles have their lengths marked on them. The area of the triangle PP_1P_2 is equal to the area of the whole rectangle less the sum of the areas of the shaded triangles. Hence

$$PP_1P_2 = a_1b_2 - \tfrac{1}{2}a_1b_1 - \tfrac{1}{2}a_2b_2 - \tfrac{1}{2}(a_1 - a_2)(b_2 - b_1) = \tfrac{1}{2}(a_1b_2 - a_2b_1).$$

This result is shown for a particular figure. By drawing other figures it will be found that the result is always correct if the angle from PP_1 to PP_2 is positive (that is, drawn in the counter-clockwise direction). If that angle is negative the formula gives the negative of the area.

Exercises

1. If the vectors AB and CD are equal show that AC and BD are equal.

2. If $AB = A_1B_1$, $BC = B_1C_1$, show that $AC = A_1C_1$.

3. The components of a vector are a, b. Show that the components of its negative are $-a$, $-b$.

4. Show that the vector $[a, b]$ is equal to the vector from the origin to the point (a, b).

5. Construct vectors equal to $[2, 3]$, $[-2, 3]$, $[-2, -3]$, and $[2, -3]$.

6. Show that the points $P\ (-1, 2)$, $Q\ (1, -2)$, $R\ (3, 4)$, $S\ (5, 0)$ are the vertices of a parallelogram.

7. A vector equal to $[-3, 4]$ begins at the point $(1, -2)$. What are the coördinates of its end?

8. The points $(1, 2)$, $(-2, -1)$, $(3, -2)$ are the vertices of a triangle. Find the vectors from the vertices to the middle points of the opposite sides.

9. Given $A\ (2, 3)$, $B\ (-4, 5)$, $C\ (-2, 3)$, find D such that $AB = CD$.

10. The middle point of a certain segment is $(1, 2)$ and one end is $(-3, 5)$. Find the coördinates of the other end.

11. By showing that the area of the triangle ABC is zero show that the points $A\ (0, -2)$, $B\ (1, 1)$ and $C\ (3, 7)$ lie on a line.

12. Find the area of the quadrilateral whose vertices are the points $(-2, -3)$, $(1, -2)$, $(3, 4)$, $(-1, 5)$.

13. Show that the vectors $[a_1, b_1]$, $[a_2, b_2]$ are parallel if and only if $a_1/a_2 = b_1/b_2$. In this way show that the line through $(-1, 2)$ and $(3, 4)$ is parallel to that through $(1, -1)$ and $(9, 3)$.

Art. 16. Multiple of a Vector

If u is a vector and r a real number, the symbol ru is used to represent a vector r times as long as u and having the same direction if r is positive, but the opposite direction if r is negative.

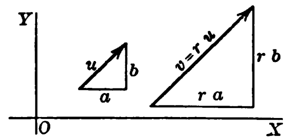

FIG. 16.

In Fig. 16, let $u = [a, b]$ and $v = ru$. Since u and v are parallel their components have lengths proportional to the lengths of u and v. Also corresponding components have the same or opposite signs according as u and v have the same or opposite directions. Hence the components of v are ra and rb. That is,

$$v = [ra, rb].$$

Hence *to multiply a vector by a number, multiply each of its components by that number.*

Example 1. Given $P_1\,(-2, 3)$, $P_2\,(1, -4)$, find a vector having the same direction as P_1P_2 but 3 times as long.

The vector required is

$$3\,P_1P_2 = 3\,[3, -7] = [9, -21].$$

Ex. 2. Find the point on the line joining $P_1\,(2, 3)$, $P_2\,(1, -2)$ and one-third of the way from P_1 to P_2.

By hypothesis $P_1P = \frac{1}{3}\,P_1P_2$, that is,

$$[x - 2, y - 3] = \tfrac{1}{3}\,[-1, -5] = [-\tfrac{1}{3}, -\tfrac{5}{3}].$$

Hence $x - 2 = -\frac{1}{3}$, $y - 3 = -\frac{5}{3}$, and consequently $x = \frac{5}{3}$, $y = \frac{4}{3}$.

Ex. 3. Show that the segment joining the points $A\,(1, -1)$, $B\,(5, 7)$ is parallel to and twice as long as that joining $C\,(4, 3)$, $D\,(2, -1)$.

The vectors are

$$AB = [4, 8], \quad CD = [-2, -4].$$

Hence $AB = -2\,CD$. Consequently the segments are parallel and the first is twice as long as the second.

Art. 17. Addition and Subtraction of Vectors

Sum of Two Vectors. — Draw a vector equal to v, beginning at the end of u. The vector from the beginning of u to the end of v is called the sum of u and v.

Let $u = [a_1, b_1]$, $v = [a_2, b_2]$. From the diagram it is seen that the components of $u + v$ are $a_1 + a_2$ and $b_1 + b_2$. This is true not

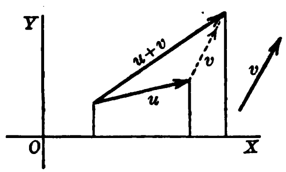

FIG. 17*a*.

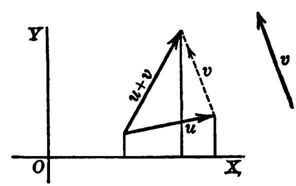

FIG. 17*b*.

only in Fig. 17*a* but also in Fig. 17*b*, for there a_2 is negative and $a_1 + a_2$ is in absolute value equal to the difference of the lengths of the projections. Hence

$$u + v = [a_1 + a_2, \quad b_1 + b_2],$$

that is, *vectors are added by adding corresponding components.*

Example 1. Show that $u + v = v + u$.

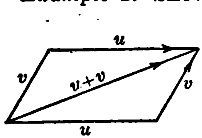

FIG. 17*c*.

The sum $u + v$ results when v is put at the end of u, while $v + u$ is given by putting u at the end of v. The two are equal since both are equal to the vector diagonal of the parallelogram whose sides are u and v (Fig. 17*c*).

Ex. 2. Show that $(u + v) + w = u + (v + w)$.

In the expression $(u + v) + w$ the sum of u and v is added to w, while in $u + (v + w)$, u is added to the sum of v and w. In

Fig. 17d let $u = AB$, $v = BC$, $w = CD$. Then

$$(u + v) + w = AC + CD = AD,$$
$$u + (v + w) = AB + BD = AD.$$

The two sums are consequently equal.

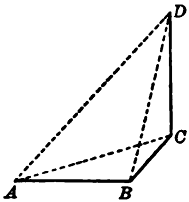

FIG. 17d.

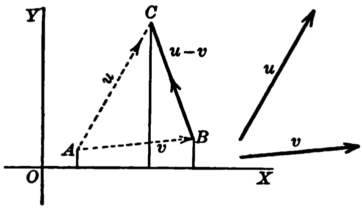

FIG. 17e.

Difference of Two Vectors.—Draw vectors equal to u and v, beginning at the same point (Fig. 17e). The vector BC from the end of v to the end of u is called the difference $u - v$.

Let $u = [a_1, b_1]$, $v = [a_2, b_2]$. It is seen from the diagram that the components of $u - v$ are $a_2 - a_1$ and $b_2 - b_1$. Consequently,

$$u - v = [a_2 - a_1, \quad b_2 - b_1],$$

that is, *vectors are subtracted by subtracting corresponding components.*

Example 1. Show that $u + (-v) = u - v$.

In Fig. 17e, $-v = BA$.

Hence

$$u + (-v) = AC + BA = BA + AC = BC = u - v.$$

Ex. 2. Two segments equal to $[2, -3]$ and $[3, -1]$ extend from the point A $(2, 2)$. Find their ends and the vector connecting their ends.

In Fig. 17f, by hypothesis,

$$OA = [2, 2], \quad AP_1 = [2, -3], \quad AP_2 = [3, -1].$$

Hence

$$OP_1 = OA + AP_1 = [2 + 2, 2 - 3] = [4, -1],$$
$$OP_2 = OA + AP_2 = [2 + 3, 2 - 1] = [5, 1].$$

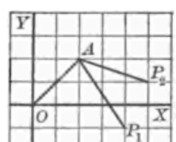

FIG. 17f.

Consequently P_1 and P_2 are $(4, -1)$ and $(5, 1)$. Also

$$P_1P_2 = AP_2 - AP_1 = [3 - 2, -1 + 3] = [1, 2].$$

Ex. 3. If weights w_1, w_2, w_3, etc., are located at the points P_1, P_2, P_3, etc., the center of gravity of these weights can be defined as the

point P satisfying the equation

$$w_1PP_1 + w_2PP_2 + w_3PP_3 + \text{etc.} = 0.$$

Find the center of gravity of three weights of 2, 3 and 5 pounds placed at the points $P_1\ (-2, 1)$, $P_2\ (1, -3)$, $P_3\ (4, 5)$ respectively.

The center of gravity P satisfies the equation

$$2\,PP_1 + 3\,PP_2 + 5\,PP_3 = 0,$$

that is,

$$\begin{aligned} 2\,[-2 - x, 1 - y] + 3\,[1 - x, -3 - y] + 5\,[4 - x, 5 - y] \\ = [19 - 10\,x, 18 - 10\,y] = 0. \end{aligned}$$

Consequently the coördinates of the center of gravity are $x = 1.9$ and $y = 1.8$.

Exercises

1. Given $P\ (1, -3)$, $Q\ (7, 1)$, $R\ (-1, 1)$, $S\ (2, 3)$, show that PQ has the same direction as RS and is twice as long.

2. Given $P_1\ (2, -3)$, $P_2\ (-1, 2)$, find the point on P_1P_2 which is twice as far from P_1 as from P_2. Also find the point on P_1P_2 produced which is twice as far from P_1 as from P_2.

3. Find the points P and Q on the line through $P_1\ (2, -1)$, $P_2\ (-4, 5)$ if $P_1P = -\frac{2}{3}\,PP_2$, $P_1Q = -\frac{2}{3}\,P_1P_2$.

4. One end of a segment is $(2, -5)$ and a point one-fourth of the distance to the other end is $(-1, 4)$. Find the coördinates of the other end.

5. Given the three points $A\ (-3, 3)$, $B\ (3, 1)$, $C\ (6, 0)$ on a line, find the fourth point D on the line such that $AD : DC = -\,AB : BC$.

6. Show that the line through $(-4, 5)$, $(-2, 8)$ is parallel to that through $(3, -1)$, $(9, 8)$.

7. If the vectors AC and AB satisfy the equation $AC = rAB$, where r is a real number, show that A, B, C lie on a line. In this way show that $(2, 3)$, $(-4, -7)$ and $(5, 8)$ lie on a line.

8. Given $A\ (1, 1)$, $B\ (2, 3)$, $C\ (0, 4)$, find the point D such that $BD = 2\,DC$. Show that $AD = \frac{1}{3}\,[AB + 2\,AC]$.

9. In Ex. 8 find the point P such that $PA + PB + PC = 0$.

10. If v is any vector let v^2 mean the square of the length of v. Given $v_1 = [2, 5]$, $v_2 = [10, -4]$, show that $(v_1 + v_2)^2 = v_1^2 + v_2^2$ and consequently that v_1 and v_2 are perpendicular to each other.

11. Show that the vectors from the vertices of a triangle to the middle points of the opposite sides have a sum equal to zero.

12. Find the center of gravity of two weights of 1 and 4 pounds placed at the points $(3, -4)$ and $(2, 7)$ respectively.

13. Show that the center of gravity of four equal weights placed at the vertices of a quadrilateral is the middle point of the segment joining the middle points of two opposite sides of the quadrilateral.

Art. 18. Slope of a Line

Let $P_1\ (x_1, y_1)$, $P_2\ (x_2, y_2)$ be two points of the line MN. The ratio

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{18a}$$

is called the *slope* of the line MN. The same value of the slope is obtained whatever *pair* of points on the line is used. For, if

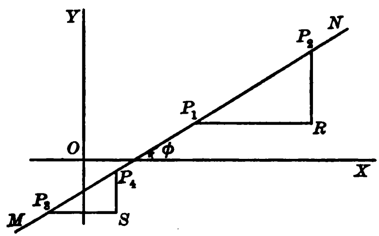

FIG. 18*a*.

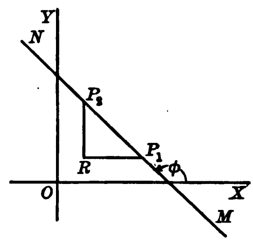

FIG. 18*b*.

P_1, P_2 and P_3, P_4 are pairs of points on the same line, the triangles P_1RP_2 and P_3SP_4 are similar and

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{RP_2}{P_1R} = \frac{SP_4}{P_3S} = \frac{y_4 - y_3}{x_4 - x_3}.$$

Let ϕ be the angle measured from the positive direction of the x-axis to the line MN. This angle is considered positive when measured in the counter-clockwise direction, negative when measured in the other direction. The tangent of this angle is RP_2/P_1R. Accordingly

$$m = \tan \phi. \tag{18b}$$

Hence the slope of a line is equal to the tangent of the angle from the positive direction of the x-axis to the line.

Since the x-axis is horizontal, the steeper the line the nearer is the

angle to 90° and consequently the greater the slope. The slope is thus a measure of the steepness of the line.

If the line extends upward to the right, as in Fig. 18*a*, the components P_1R and RP_2 have the same sign and the slope is positive. If the line extends upward to the left, as in Fig. 18*b*, the components have opposite signs and the slope is negative.

Parallel Lines. — *If two lines are parallel, they have the same slope;* for the angles ϕ_1 and ϕ_2 (Fig. 18c) are then equal and their slopes, $\tan \phi_1$ and $\tan \phi_2$, are equal. Conversely, if the slopes are equal, the angles are equal and the lines are parallel.

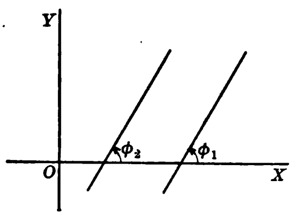

FIG. 18c.

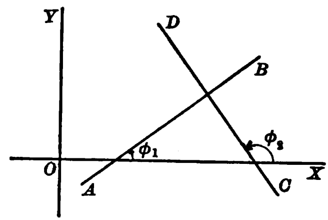

FIG. 18*d*.

Perpendicular Lines. — In Fig. 18*d*, let the lines AB and CD be perpendicular. Since the exterior angle is equal to the sum of the two opposite interior angles

$$\phi_2 = \phi_1 + 90°.$$

Consequently
$$\tan \phi_2 = -\cot \phi_1 = -\frac{1}{\tan \phi_1}.$$

Conversely, if this relation holds, ϕ_1 and ϕ_2 differ by 90° and the lines are perpendicular. If m_1 and m_2 are the slopes, $m_1 = \tan \phi_1$, $m_2 = \tan \phi_2$, and

$$m_2 = -\frac{1}{m_1}. \qquad (18c)$$

Two lines are perpendicular if and only if the slope of one is equal to the negative reciprocal of the slope of the other.

Angle between Two Lines. — Let two lines L_1 and L_2 make with the x-axis the angles ϕ_1 and ϕ_2 (Fig. 18*e*). If β is the angle from L_1 to

L_2 (positive when measured in the counter-clockwise direction), then

$$\beta = \phi_2 - \phi_1.$$

Hence

$$\tan \beta = \tan (\phi_2 - \phi_1)$$
$$= \frac{\tan \phi_2 - \tan \phi_1}{1 + \tan \phi_1 \tan \phi_2}.$$

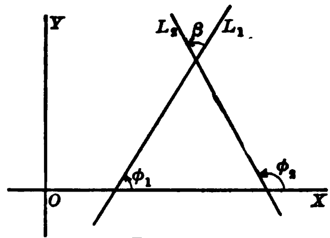

FIG. 18e.

If m_1 and m_2 are the slopes of the lines, $m_1 = \tan \phi_1$, $m_2 = \tan \phi_2$, whence

$$\tan \beta = \frac{m_2 - m_1}{1 + m_1 m_2}. \tag{18d}$$

In using this formula it should be remembered that β is measured from L_1 to L_2.

Example 1. Show that the line through A $(1, -3)$, B $(5, -1)$ is parallel to that through C $(-3, -2)$, D $(-1, -1)$.

$$\text{Slope of } AB = \frac{-1+3}{5-1} = \frac{1}{2},$$
$$\text{slope of } CD = \frac{-1+2}{-1+3} = \frac{1}{2}.$$

Since the slopes are equal the lines are parallel.

Ex. 2. Show that the line through A $(2, 3)$, B $(-4, 5)$ is perpendicular to that through C $(1, 1)$, D $(2, 4)$.

$$\text{Slope of } AB = \frac{5-3}{-4-2} = -\frac{1}{3},$$
$$\text{slope of } CD = \frac{4-1}{2-1} = 3.$$

Since either slope is the negative reciprocal of the other, the lines are perpendicular.

Ex. 3. Find the angles of the triangle formed by the points A $(-1, 2)$, B $(1, 1)$, C $(3, 4)$.

The triangle is shown in Fig. 18*f*. The slopes of AB, BC, CA are found to be $-\frac{1}{2}$, $\frac{3}{2}$ and $\frac{1}{2}$ respectively. Between two lines such as AB and AC are two positive angles less than 180°. One of

these is interior, the other exterior to the triangle. From the figure the positive interior angle A is seen to extend from AB to AC. Consequently,

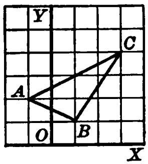
FIG. 18f.

$$\tan A = \frac{\frac{1}{2} - (-\frac{1}{2})}{1 + \frac{1}{2}(-\frac{1}{2})} = \frac{4}{3}.$$

In the same way it is found that $\tan B = -8$, $\tan C = \frac{4}{7}$. The angles of the triangle are therefore the positive angles less than 180° whose tangents are $\frac{4}{3}$, -8 and $\frac{4}{7}$. The angles A and C are acute, B is obtuse.

Exercises

1. Find the slopes of the sides of the triangle formed by the points $(-1, 2)$, $(1, 3)$ and $(2, -4)$.

2. Find the angle from the x-axis to the line joining $(1, 1)$ and $(-4, 5)$.

3. The angles from the x-axis to three lines are 45°, 120° and $-30°$ respectively. Find the slopes of the lines.

4. The sides of a triangle have slopes equal to $\frac{1}{2}$, 1 and 2. Show that the triangle is isosceles.

5. If the slopes of AB and BC are equal, show that A, B, C lie on a line. In this way show that the points $(4, 1)$, $(1, 2)$ and $(-5, 4)$ lie on a line.

6. Show that the line through $(1, -3)$ and $(-1, 3)$ is parallel to that through $(3, -5)$ and $(0, 4)$.

7. Show that the lines determined by the pairs of points $(2, 3)$, $(3, 5)$ and $(1, 7)$, $(3, 6)$ are perpendicular to each other.

8. Show that the vectors $[a, b]$ and $[c, d]$ are parallel if $b/a = d/c$ and perpendicular if $b/a = -c/d$.

9. Find the interior angles of the triangle formed by the points $(2, 3)$, $(-4, 5)$ and $(1, -2)$.

10. Lines join the point $(-2, 2)$ to the points $(-3, 1)$, $(0, 2)$ and $(1, -1)$. Show that one of these bisects the angle between the other two.

11. Two lines have slopes equal to 2 and -3. Find the slopes of the two bisectors of the angles between these lines.

12. Find the angle between the lines joining the origin to the two points of trisection of the segment joining $(-2, 3)$ and $(1, -7)$.

13. The slope of AB is $\frac{1}{2}$. Find the slope of CD if the angle from AB to CD is 30°.

14. By showing that the angles CAD and CBD are equal show that the points A (6, 11), B (−11, 4), C (−4, −13), D (1, −14) lie on a circle.

15. A point is 7 units distant from the origin and the slope of the line joining it to (3, 4) is $\frac{1}{2}$. Find its coördinates.

16. A point is equidistant from (2, 1) and (−4, 3) and the slope of the line joining it to (1, −1) is $\frac{2}{3}$. Find its coördinates.

Art. 19. Graphs

In many cases corresponding values of two related quantities are known. Each pair of values can be taken as the coördinates of a point. The totality, or locus, of such points is called a *graph.* This graph exhibits pictorially the relation of the quantities represented. There are three cases differing in the accuracy with which intermediate values are known.

Statistical Graphs. — Sometimes definite pairs of values are given but there is no information by which intermediate values can be even approximately inferred. In such cases the values are plotted and consecutive points connected by straight lines to show the order in which the values are given.

Example 1. The temperature at 6 A.M. on ten consecutive days at a certain place is given in the following table:

Jan.	1	2	3	4	5	6	7	8	9	10
	19°	27°	11°	40°	34°	28°	36°	18°	42°	38°

These values are represented by points in Fig. 19a. Points representing temperatures on consecutive days are connected by straight lines. At times between those marked no estimate of the temperature can be made.

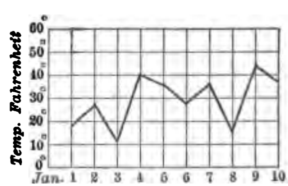

FIG. 19a.

Physical Graphs. — In other cases it is known that in the intervals between the values given the variables change nearly proportionally. In such cases the points are plotted and as smooth a curve as possible is drawn through them. Points

of this curve between those plotted are assumed to represent approximately corresponding values of the variables.

Ex. 2. The observed temperatures θ of a vessel of cooling water at times t, in minutes, from the beginning of observation were

$t = 0$	1	2	3	5	7	10	15	20
$\theta = 92°$	85.3°	79.5°	74.5°	67°	60.5°	53.5°	45°	39.5°

These values are plotted in Fig. 19*b*. Since the temperature decreases gradually and during the intervals given at nearly constant rates a smooth curve drawn through these points will represent approximately the relation of temperature and time throughout the experiment.

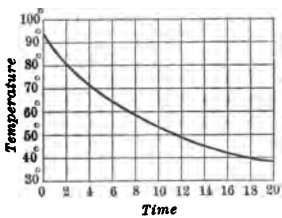

FIG. 19*b*.

The Graph of an Equation. — In other cases the equation connecting the variables is known. The graph then consists of all points whose coördinates satisfy the equation. Arbitrary values are assigned to either of the variables, the corresponding values of the other variable calculated, and the resulting points plotted. When the points have been plotted so closely that between consecutive points the curve is nearly straight, a smooth curve is drawn through them.

Ex. 3. Plot the graph of the equation $y = x^2$.

In the following table values are assigned to x and the corresponding values of y calculated:

$x =$	-5	-4	-3	-2	-1	0	1	2	3	4	5
$y =$	25	16	9	4	1	0	1	4	9	16	25

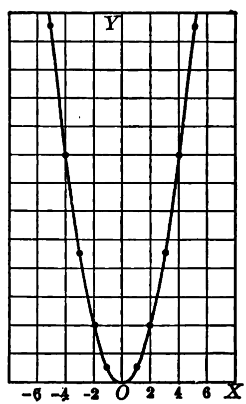

FIG. 19*c*.

The points are plotted in Fig. 19*c*. The part of the curve shown extends from $x = -5$ to $x = +5$. The whole curve extends to an indefinite distance. Horizontal lines cut the

the curve at equal distances right and left of the y-axis. This is expressed by saying that the curve is symmetrical with respect to the y-axis.

Ex. 4. The force of gravitation between two bodies varies inversely as the square of their distance apart, that is, $F = k/d^2$, k being constant. Assuming $k = 10$, plot the curve representing the relation of force and distance.

In the following table values are assigned to d and the corresponding values of F calculated:

$d = 0$	0.5	1	1.5	2	3	5	10
$F = \infty$	40	10	4.4	2.5	1.1	0.4	0.1

The distance d cannot be negative. For very large values of d, F is very small. Hence when d is large the curve very nearly coincides with the horizontal axis. When d is very small F is very large and the curve very nearly coincides with the vertical axis (Fig. 19d).

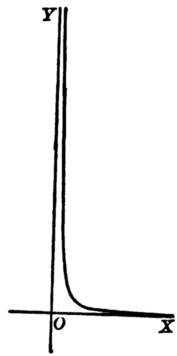

FIG. 19d.

Units.— Before a graph can be plotted a length must be chosen on each axis to represent a unit measure of the quantity represented by that coördinate. If the quantities represented by x and y are of different kinds, for example temperature and time, these lengths can be chosen independently. If, however, these quantities are of the same kind it is usually more convenient to have the same unit of length along both axes. This is also the case in graphing mathematical equations where no physical interpretation is given to x and y. Other things being equal, a large curve is better than a small one. Such units of length should then be chosen as to make the curve spread both vertically and horizontally very nearly over the region available for it. If several graphs are to be compared, the same unit lengths should be used for all.

Exercises

1. The price of steel rails each year from 1892 to 1907 was as follows, in dollars per gross ton:

30 28 24 24 28 19 18 28 32 27 28 28 28 28 28 28

Make a graph showing the relation of year and price. (Let x be the

number of years beyond 1892 and y the amount that the price exceeds 18 dollars.)

2. The amplitudes of successive vibrations of a pendulum set in motion and left free were

Number of vibration	1	2	3	4	5	6	7
Amplitude	69	48	33.5	23.5	16.5	11.5	8

Plot points representing these pairs of values and draw a smooth curve through them. Do points of this curve between those plotted have any physical meaning?

3. The atomic weight, W, and specific heat, S, of several chemical elements are shown in the following table:

$W =$	7	9.1	11	12	23	28	39	55	56	108	196
$S =$	0.94	0.41	0.25	0.147	0.29	0.177	0.166	0.122	0.112	0.057	0.032

Make a graph showing the relation of specific heat and atomic weight.

4. The table below gives the maximum length of spark between needle points of an alternating current under standard conditions of needles, temperature and barometric pressure.

L = length in millimeters. V = electromotive force in kilovolts.

$V =$	10	15	20	25	30	35	40	45	50	60	70	80
$L =$	12	18	25	33	41	51	62	75	90	118	149	180

Make a curve showing the relation of voltage and length of spark.

5. In the table below are given the maximum vapor pressures of water at various temperatures, where T = temperature in degrees Centigrade and P = pressure in centimeters of mercury:

$T =$	0	10	20	30	40	50	60	70	80	90	100
$P =$	0.46	0.91	1.74	3.15	5.49	9.20	14.9	23.3	35.5	52.5	76

Make a graph showing the relation of temperature and vapor pressure.

6. The rate for letter postage is two cents for each ounce or fraction. Make a graph showing the relation of weight and cost of postage. (The cost for 1.5 ounces is the same as that for 2, etc.)

7. Make a graph showing the number of centimeters y in x inches.

8. Make a graph showing the relation between the side and the area of a square.

9. According to Boyle's law the pressure and volume of a gas at constant temperature are connected by the equation pv = constant. Assuming the constant equal to 10, construct the curve representing the relation of pressure and volume. Can p or v be negative?

10. Plot the curve $y = x^2 - 2x + 3$. Show that it passes through the point (1, 2).

11. On the same diagram plot the curves $y = x^4$ and $y = (x + 1)^4$. How are they related? Show that they both pass through the point $(-\frac{1}{2}, \frac{1}{16})$.

12. On the same diagram plot the curves $x = 1/y^2$ and $x = 1/(y-2)^2$. How are they related? Show that both pass through (1, 1).

13. On the same diagram plot the graphs of $x + y = 2$ and $2x - 3y = 1$. What are they? Find their point of intersection. (It must have coördinates satisfying both equations.)

Art. 20. Equation of a Locus

If a locus and an equation are such that (1) *every point on the locus has coördinates that satisfy the equation and* (2) *every point whose coördinates satisfy the equation lies on the locus, then the equation is said to represent the locus and the locus to represent the equation.*

Usually a locus is defined by a property possessed by each of its points. Thus a circle is the locus of points at a constant distance from a fixed point. To find the equation of a locus express this property by means of an equation connecting the coördinates of each locus point. The graph is constructed by using the definition or by plotting from the equation.

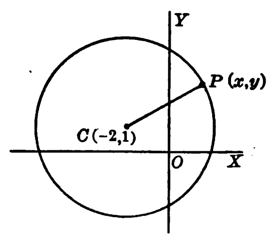

FIG. 20a.

Example 1. Find the equation of a circle with center C $(-2, 1)$ and radius equal to 4.

Let P (x, y) (Fig. 20a) be any point on the circle. By the definition of a circle, $CP = 4$, and this is equivalent to

$$\sqrt{(x + 2)^2 + (y - 1)^2} = 4,$$

which is the equation required. By squaring this can be reduced to the form

$$(x + 2)^2 + (y - 1)^2 = 16.$$

Ex. 2. A curve is described by a point P (x, y) moving in such a way that its distance from the x-axis equals its distance from the point Q (1, 1). Construct the curve and find its equation.

The curve is constructed by locating points such that $MP = QP$. Since no point below the x-axis can be equally distant from Q and the x-axis, the curve lies entirely above the x-axis. Hence y is positive and equal to the distance MP. Also $QP = \sqrt{(x-1)^2 + (y-1)^2}$. Consequently, if $P\ (x, y)$ is any point on the curve,

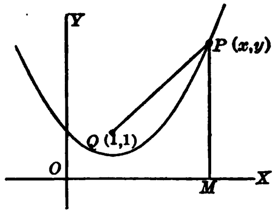

FIG. 20*b*.

$$y = \sqrt{(x-1)^2 + (y-1)^2}.$$

Conversely, if this equation is satisfied, the point P is equidistant from Q and the x-axis and so lies on the curve. It is therefore the equation of the curve. Since y cannot be negative the equation is equivalent to

$$y^2 = (x-1)^2 + (y-1)^2,$$

and consequently to

$$x^2 - 2x - 2y + 2 = 0.$$

Exercises

1. What loci are represented by the equations, (a) $x = 3$, (b) $y = -2$, (c) $y = 2x$, (d) $x^2 + y^2 = 1$?

2. The point $P\ (x, y)$ is equidistant from $(1, 2)$ and $(3, -4)$. What is the locus of P? Find its equation.

3. The point $P\ (x, y)$ is twice as far from the x-axis as from the y-axis. What is the locus of P? (Two parts.) Find its equation.

4. The slope of the line joining $P\ (x, y)$ to $(-2, 3)$ is equal to 3. What is the locus of P? Find its equation.

5. Given $A\ (-3, 1)$, $B\ (2, 0)$, $P\ (x, y)$. If the slopes of AB and BP are equal what is the locus of P? Find its equation.

6. The point $P\ (x, y)$ is equidistant from the y-axis and the point $(-3, 4)$. Construct the locus of P. Find its equation.

7. Given $A\ (2, 3)$, $B\ (-1, 1)$, $C\ (2, -3)$, $P\ (x, y)$. If P moves along the line through A perpendicular to BC, show that

$$PC^2 - PB^2 = AC^2 - AB^2.$$

Find the equation of the line described by P.

8. Find the equation of a circle with center $(-1, 2)$ and radius 5.

9. Find the equation of the circle whose diameter is the segment joining $(2, -3)$ and $(-1, 4)$.

10. Given $P_1\,(1, 3)$, $P_2\,(4, -1)$, find the equation of the locus described by $P\,(x, y)$ if the sum of its distances from P_1 and P_2 is 5. Free the equation of radicals and show that the result is

$$(4\,x + 3\,y - 13)^2 = 0.$$

Make a graph of this equation. Do all points of this graph belong to the locus?

11. Given $A\,(2, -2)$, $B\,(6, 0)$, $C\,(7, 3)$, $P\,(x, y)$. If the angle from AB to AC is equal to the angle from PB to PC, show that the locus of P is the circle through A, B, C. Calculate the tangents of the two angles, equate them and so get the equation of the circle.

Art. 21. Point on a Locus

If a point lies on a locus its coördinates satisfy the equation of the locus. Hence to ascertain whether a point lies on a given locus, substitute its coördinates in the equation of the locus and find out whether the equation is satisfied.

Example 1. Show that the curve $x^2 + y^2 = 2\,x$ passes through the origin.

Substituting the values $x = 0$, $y = 0$ the equation becomes $0 = 0$. Since the equation is satisfied the curve passes through the origin. In the same way it could be shown that any locus, represented by a polynomial equation with no constant term, passes through the origin.

Ex. 2. If the curve $x^2 + y^2 = 2\,ax$ passes through the point $(2, -1)$ find the value of the constant a.

The coördinates 2, -1 must satisfy the equation. Hence

$$2^2 + (-1)^2 = 2\,a\,(2).$$

Consequently, $a = \frac{5}{4}$.

Ex. 3. If the locus of $y = mx + b$ passes through $(-1, 1)$ and $(3, 2)$, find the values of m and b.

The conditions required are

$$1 = -m + b, \quad 2 = 3\,m + b.$$

The solution of these equations is $m = \frac{1}{4}$, $b = \frac{5}{4}$.

Intersection of Curves. — An intersection of two curves must have coördinates satisfying both equations. Hence to find the points of

intersection, solve the equations simultaneously. All solutions should be checked by substitution.

Example 1. Plot the curves represented by the equations $y^2 = 2x$ and $x^2 = 2y$ and find their points of intersection.

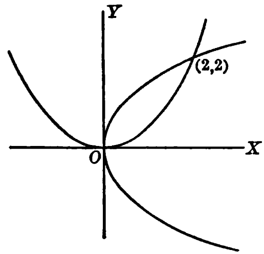

FIG. 21*a*.

The curves are shown in Fig. 21*a*. From the figure it is seen that there are two real points of intersection. Squaring the first equation and substituting the value of x^2 from the second,

$$y^4 = 4x^2 = 8y.$$

Consequently, $y(y^3 - 8) = 0$. The two real solutions are $y = 0, 2$. The corresponding values of x are $x = \frac{1}{2}y^2 = 0, 2$. The points of intersection are then (0, 0) and (2, 2). Substitution shows that the coördinates of these points satisfy both equations.

Ex. 2. Plot the curves $xy = 2$, $x^2 + y^2 = 4x$, and find their points of intersection.

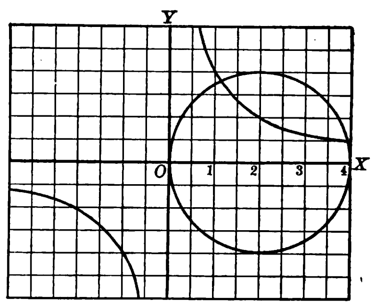

FIG. 21*b*.

The curves are shown in Fig. 21*b*. It is seen that there are two points of intersection. Elimination of y gives

$$x^4 - 4x^3 + 4 = 0.$$

The equation can only be solved approximately. The figure indicates that the solutions are near $x = 1.2$ and $x = 4$. By substitution the following values are found:

$x =$	1.1	1.2	3.9	4
$x^4 - 4x^3 + 4 =$	0.14	−0.84	−1.9	4

There is a root between 1.1 and 1.2 and another between 3.9 and 4. When $x = 1.15$ the expression $x^4 - 4x^3 + 4$ is found to be negative.

Hence, to one decimal, the value 1.1 is a root. Similarly 3.9 is found to be the other root. The corresponding values of y are $2/x = 1.8$ and 0.6. The points of intersection are then approximately

$$(1.1,\quad 1.8),\quad (3.9,\quad 0.6).$$

Ex. 3. Plot the loci of the three equations

$$x + y = 1, \quad 2x + 3y = 5, \quad x^2 + y^2 = 13,$$

and show that they pass through a point.

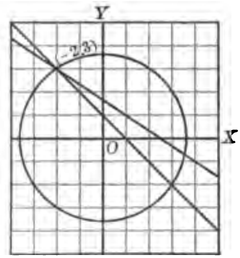

FIG. 21c.

The graphs are shown in Fig. 21c. The solution of the first two equations is $x = -2$, $y = 3$. These values satisfy the third equation. Consequently all the loci pass through the point $(-2, 3)$.

Art. 22. Tangent Curves

If at a point of intersection two curves have the same direction, they are called tangent. Let the curves AB and CD be tangent at P (Fig. 22a). CD can be considered as the limiting position of a curve $C'D'$ cutting AB in two points P_1 and P_2 close together. The equations of AB and $C'D'$ have two simultaneous solutions that are nearly the same. As $C'D'$ approaches CD these solutions approach equality. One might then expect that if the equations of AB and CD are solved simultaneously two of the solutions will be equal and that, conversely, equal roots occurring in the simultaneous solution of two equations indicate tangency. This should however be checked by the graphs as there are other circumstances that result in coincident solutions.

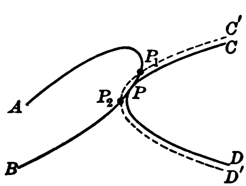

FIG. 22a.

Example 1. Show that the line $x + y = 2$ and the circle $x^2 + y^2 = 2$ are tangent and find the point of tangency.

Eliminating y,

$$x^2 + (2 - x)^2 - 2 = 0.$$

This equation is equivalent to $2(x-1)^2 = 0$ and so has two equal roots $x = 1$. The corresponding value of y is $2 - x = 1$. The line and circle intersect in only one point (1, 1). They must therefore be tangent at (1, 1).

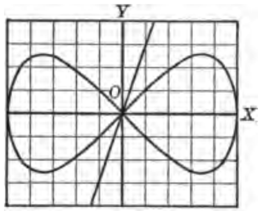

FIG. 22*b*.

Ex. 2. Plot the graphs of $y^2 = x^2 - x^4$ and $y = 3x$ and find their intersections.

The graphs are shown in Fig. 22*b*. The first is a curve like a horizontal figure 8. The second is a straight line through the origin. Eliminating y,

$$x^4 + 8x^2 = 0.$$

This equation has two roots $x = 0$. Two parts of the curve pass through the origin and both cut the line at that point. The curve and line are not however tangent.

Exercises

1. A, B, C, D, E being any fixed numbers, show that the curve whose equation is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = 0$$

passes through the origin.

2. If x_1, y_1, m are constant show that the locus of the equation

$$y - y_1 = m(x - x_1)$$

passes through the point (x_1, y_1).

3. The curve $x^2/a^2 + y^2/b^2 = 1$ passes through the points (0, 1) and (2, 0). Find the values of a and b and plot the curve.

4. The curve $y^2 = ax + b$ passes through the points (0, -1) and (1, 2). Find the values of a and b and plot the curve.

Plot the following pairs of loci and find their points of intersection:

5. $3x + 2y = 1$, $2x - y = 0$.
6. $x^2 + y^2 = 10$, $2y - 3x = 3$.
7. $x^2 - y^2 = 3$, $x^2 + xy + y^2 = 7$.
8. $y^2 = x + 1$, $xy = 2$.
9. $xy^2 = 1$, $2x - y = 3$.
10. $x^2 - 2y^2 = 7$, $2y + 3x = 7$.

11. $2x^2 + y^2 = 6,$
$x - y = 1.$

12. $x^2y + 3y = 4,$
$x + 2y = 3.$

13. $y = 2x^3,$
$y = 3x^2 - 5.$

14. $x^2 + 4y = 0,$
$x^2 + y^2 + 6x + 7 = 0.$

15. Show that the following curves have a common point:

$$x^2 + y^2 = 10, \quad y^2 = x + 4, \quad y = \frac{2}{x+5}.$$

CHAPTER 3

STRAIGHT LINE AND CIRCLE

Art. 23. Equation of a Straight Line

Let P_1 (x_1, y_1) be a fixed point and P (x, y) a variable point on the line MN. If the line is not perpendicular to the x-axis (Fig. 23a), let its slope be m. Since the line passes through P_1 and P, by the definition of slope,

$$m = \frac{y - y_1}{x - x_1}.$$

This is an equation satisfied by the coördinates, x and y, of any point P on the line. Conversely, if the coördinates of any point P (x, y) satisfy this equation, the slope of P_1P is m and consequently P lies on MN. It is therefore the equation of MN. It can be written

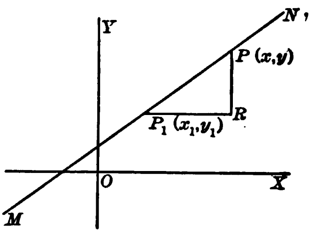

FIG. 23a.

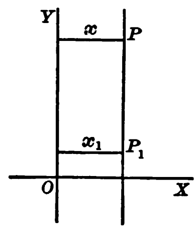

FIG. 23b.

$$y - y_1 = m(x - x_1). \tag{23a}$$

This is consequently the equation of a line through (x_1, y_1) *with slope* m.

Let the line MN cross the y-axis at $(0, b)$. Replacing x_1, y_1 in equation (23a) by 0, b, the equation becomes

$$y = mx + b. \tag{23b}$$

The quantity b is called the intercept of the line on the y-axis. It is

numerically equal to the distance along the y-axis to the line, is positive when the line is above the origin and negative when it is below. Equation (23b) represents the line with slope m and intercept b on the y-axis.

If the line is perpendicular to the x-axis (Fig. 23b) its slope is infinite and equations (23a) and (23b) cannot be used. In this case the figure shows that

$$x = x_1. \tag{23c}$$

Conversely, if the abscissa of a point is x_1, it lies on the line. Therefore (23c) is the equation of a line through (x_1, y_1) perpendicular to the x-axis.

Example 1. Find the equation of a line through $(-1, 2)$, the angle from the x-axis to the line being 30°.

The slope of the line is

$$m = \tan (30°) = \tfrac{1}{3} \sqrt{3}.$$

The equation of the line is then

$$y - 2 = \tfrac{1}{3} \sqrt{3}\,(x + 1).$$

Ex. 2. Find the equation of the line through the points (2, 0) and (1, −3).

The slope of the line is

$$\frac{-3 - 0}{1 - 2} = +3.$$

Since the line passes through (2, 0) and has a slope equal to 3, its equation is $y - 0 = 3\,(x - 2)$, whence

$$y = 3\,x - 6.$$

Ex. 3. Find the equation of the line through (1, −1) perpendicular to the line through (2, 3) and (3, −2).

The slope of the line through (2, 3) and (3, −2) is −5. The slope of a perpendicular line is $-1/(-5) = \frac{1}{5}$. The equation of the line with this slope passing through (1, −1) is

$$y + 1 = \tfrac{1}{5}\,(x - 1),$$

which is the equation required.

Ex. 4. Show that the equation $2x - 3y = 5$ represents a straight line. Find its slope.

Solving for y,

$$y = \tfrac{2}{3}x - \tfrac{5}{3}.$$

Comparing this with the equation $y = mx + b$, it is seen that the two are equivalent if $m = \frac{2}{3}$, $b = -\frac{5}{3}$. Therefore the given equation represents a straight line with slope $\frac{2}{3}$ and intercept $-\frac{5}{3}$ on the y-axis.

Exercises

1. The angle from the x-axis to a line is 60°. The line passes through $(-1, -3)$. Find its equation.
2. Find the equation of the line through $(2, -1)$ and $(3, 2)$.
3. Find the equation of the line through $(2, 3)$ and $(2, -4)$.
4. Find the equation of the line through $(1, 2)$ parallel to the x-axis.
5. Find the equation of the perpendicular bisector of the segment joining $(-3, 5)$ and $(-4, 1)$.
6. An equilateral triangle has its base in the x-axis and its vertex at $(3, 5)$. Find the equations of its sides.
7. A line is perpendicular to the segment between $(-4, -2)$ and $(2, -6)$ at the point one-third of the way from the first to the second point. Find its equation.
8. Find the equation of the line through $(3, 5)$ parallel to that through $(2, 5)$ and $(-5, -2)$.
9. One diagonal of a parallelogram joins the points $(4, -2)$ and $(-4, -4)$. One end of the other diagonal is $(1, 2)$. Find its equation and length.
10. A diagonal of a square joins the points $(1, 2)$, $(2, 5)$. Find the equations of the sides of the square.
11. The base of an isosceles triangle is the segment joining $(-2, 3)$ and $(3, -1)$. Its vertex is on the y-axis. Find the equations of its sides.
12. Show that the equation $2x - y = 3$ represents a straight line. Find its slope and construct the line.
13. Show that the equations

$$2x + 3y = 5, \quad 3x - 2y = 7$$

represent two perpendicular straight lines.

14. Perpendiculars are dropped from the point $(5, 0)$ upon the sides of the triangle whose vertices are the points $(4, 3)$, $(-4, 3)$, $(0, -5)$. Show that the feet of the perpendiculars lie on a line.

Art. 24. First Degree Equation

Any straight line is represented by an equation of the first degree in rectangular coördinates. In fact, if the line is not perpendicular to the x-axis, its equation has been shown to be

$$y - y_1 = m(x - x_1),$$

x_1, y_1 and m being constants. If it is perpendicular to the x-axis its equation is

$$x = x_1.$$

Since both of these equations are of the first degree in x and y, it follows that any straight line has an equation of the first degree in rectangular coördinates.

Conversely, *any equation of the first degree in rectangular coördinates represents a straight line.* For any equation of the first degree in x and y has the form

$$Ax + By + C = 0, \tag{24}$$

A, B, C being constant. If B is not zero, this equation is equivalent to

$$y = -\frac{A}{B}x - \frac{C}{B},$$

which represents a line with slope $-A/B$ and intercept $-C/B$ on the y-axis. If B is zero, the equation is equivalent to

$$x = -\frac{C}{A},$$

which represents a line perpendicular to the x-axis passing through the point $(-C/A, 0)$. Hence in any case an equation of the first degree represents a straight line.

Graph of First Degree Equation. — Since a first degree equation represents a straight line, its graph can be constructed by finding two points and drawing the straight line through them. The best points for this purpose are usually the intersections of the line and coördinate axes. The intersection A (Fig. 24a) with the x-axis is found by letting $y = 0$ and solving the equation of the line for the

corresponding value of x. Similarly, the intersection B with the y-axis is found by letting $x = 0$ and solving for y. The abscissa of

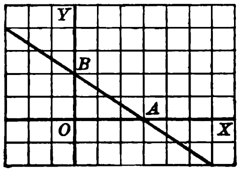

FIG. 24a.

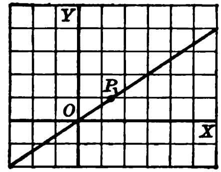

FIG. 24b.

A and the ordinate of B are called the *intercepts* of the line on the coördinate axes.

If the line passes through the origin (Fig. 24b) the points A and B coincide at the origin and it is necessary to find another point on the line. This is done by assigning any value to one of the coördinates and calculating the resulting value of the other coördinate.

Example 1. Plot the line $2x + 3y = 6$ and find its intercepts on the axes.

Substituting $y = 0$ gives $x = 3$ and substituting $x = 0$ gives $y = 2$. Hence the line passes through the points A (3, 0) and B (0, 2). Its intercept on the x-axis is 3 and on the y-axis 2.

Ex. 2. Construct the line whose equation is $2x - 3y = 0$.

When x is zero y is zero. The line then passes through the origin. When $y = 1$, $x = \frac{3}{2}$. Hence the line OP_1 (Fig. 24b) through the origin and the point $(\frac{3}{2}, 1)$ is the one required.

Slope of a Line. — If the line is perpendicular to the x-axis, its slope is infinite. If it is not perpendicular to the x-axis, its equation is

$$y = mx + b.$$

The slope is the coefficient of x in this equation. Consequently, if the equation of a line is solved for y, its slope is the coefficient of x.

Example 1. Find the slope of the line $3x - 5y = 7$.

Solving for y

$$y = \frac{3}{5}x - \frac{7}{5}.$$

The slope of the line is therefore $\frac{3}{5}$.

Ex. 2. Find the angle from the line $x + y = 3$ to the line $y = 2x + 5$.

The slope of the first line is -1, that of the second 2. The angle β between the lines is determined by the equation

$$\tan \beta = \frac{2-(-1)}{1+2(-1)} = -3.$$

The negative sign signifies that the angle is negative or obtuse.

Ex. 3. Find the equation of the line through (3, 1) perpendicular to the line $2x + 4y = 5$.

The given equation can be written $y = -\frac{1}{2}x + \frac{5}{4}$. Its slope is consequently $-\frac{1}{2}$. The slope of a perpendicular line is 2. The equation of the line through (3, 1) with slope 2 is

$$y - 1 = 2(x - 3),$$

which is the equation required.

Exercises

Plot the straight lines represented by the following equations, find their slopes and intercepts:

1. $2y - 3 = 0$.
2. $5x + 7 = 0$.
3. $x + y = 2$.
4. $2x + 3y - 5 = 0$.
5. $3x - 6y + 7 = 0$.
6. $2x + 5y + 8 = 0$.
7. $4x + 3y = 0$.
8. $3x - 4y = 0$.

9. Show that the equation

$$(2x + 3y - 1)(x - 7y + 2) = 0$$

represents a pair of lines.

10. Show that $(x + 4y)^2 = 9$ represents two parallel lines.

11. Show that $x^2 = (y - 1)^2$ represents two lines perpendicular to each other.

12. Show that the lines $3x + 4y - 7 = 0$, $9x + 12y - 8 = 0$ are parallel.

13. Show that the lines $x + 2y + 5 = 0$, $4x - 2y - 7 = 0$ are perpendicular to each other.

14. Find the interior angles of the triangle formed by the lines

$$x = 0, \quad x - y + 2 = 0, \quad 2x + 3y - 21 = 0.$$

15. Find the equation of the line whose intercepts on the x and y axes are 2 and -3 respectively.

16. Find the equation of the line whose slope is 5 and intercept on the y-axis -4.

17. Find the equation of the line through $(3, -1)$ parallel to the line $x - y = 8$.

18. Find the equation of the line through $(2, -1)$ perpendicular to the line $9x - 8y + 6 = 0$.

19. Find the projection of the point $(2, -3)$ on the line $x - 4y = 5$.

20. Find the equation of the line perpendicular to $3x - 5y = 9$ and bisecting the segment joining $(-1, 2)$ and $(4, 5)$.

21. Find the lengths of the sides of the triangle formed by the lines $4x + 3y - 1 = 0$, $3x - y - 4 = 0$, $x + 4y - 10 = 0$.

22. A line passes through $(2, 2)$. Find its equation if the angle from it to $3x - 2y = 0$ is $45°$.

23. Find the equation of the line through $(4, \frac{8}{3})$ and the intersection of the lines $3x - 4y - 2 = 0$, $12x - 15y - 8 = 0$.

24. Find the equation of the line through the intersection of the lines $2x - y + 5 = 0$, $x + y + 1 = 0$, and the intersection of the lines $x - y + 7 = 0$, $2x + y - 5 = 0$.

25. Find the locus of a point if the tangents from it to two fixed circles are of equal length.

26. What angle is made with the axis of y by a straight line whose equation is $\frac{1}{4}y + \frac{1}{3}x = 1$?

Art. 25. The Expression $Ax + By + C$

At each point P of the plane a first degree expression $Ax + By + C$ has a definite value obtained by putting for x and y the coördinates of P. Thus at the point $(1, 2)$ the expression has the value $A + 2B + C$. Points where the expression is zero constitute a line whose equation is $Ax + By + C = 0$. If the point P moves slowly the value of the expression changes continuously. A number changing continuously can only change sign by passing through zero. If the point P does not cross the line the expression cannot become zero and so cannot change sign. Therefore *at all points on one side of the line $Ax + By + C = 0$ the expression $Ax + By + C$ has the same sign.*

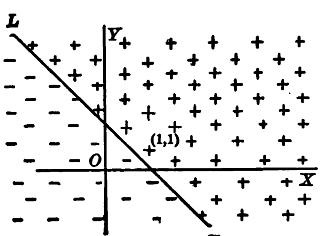

FIG. 25a.

Example 1. Determine the region in which $x + y - 1 > 0$.

The equation $x + y - 1 = 0$ represents the line LK (Fig. 25a).

At all points on one side of LK the expression then has the same sign. At (1, 1)

$$x + y - 1 = 1 + 1 - 1 = 1$$

which is positive. It is seen from the figure that (1, 1) is above LK. Hence, at all points above LK, $x + y - 1$ is positive. At the origin

$$x + y - 1 = 0 + 0 - 1 = -1$$

which is negative. The origin is below the line. Hence at all points below LK the expression $x + y - 1$ is negative. The region in which $x + y - 1 > 0$ is therefore the part of the plane above the line.

Ex. 2. Determine the region in which $x + y > 0$, $x + 2y - 2 < 0$ and $x - y - 1 < 0$.

In Fig. 25*b* the lines $x + y = 0$, $x + 2y - 2 = 0$, $x - y - 1 = 0$ are marked (1), (2), (3) respectively. Proceeding as in the last example, it is found that $x + y > 0$ above (1), $x + 2y - 2 < 0$ below (2) and $x - y - 1 < 0$ on the left of (3). Hence the three inequalities hold in the shaded triangle which is the part common to the three regions.

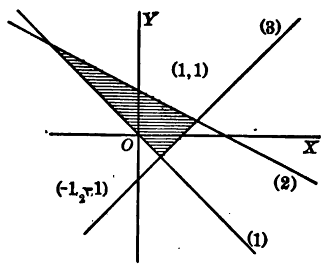

FIG. 25*b*.

Art. 26. Distance from a Point to a Line

We wish to find the distance from the point P_1 (x_1, y_1) to the line LK whose equation is $Ax + By + C = 0$. In Fig. 26*a* let MP_1 be perpendicular to the x-axis and DP_1 to the line LK. Let ϕ be the angle from OX to LK. Then

(*a*) $DP_1 = QP_1 \cos\phi$
$\qquad = (MP_1 - MQ)\cos\phi.$

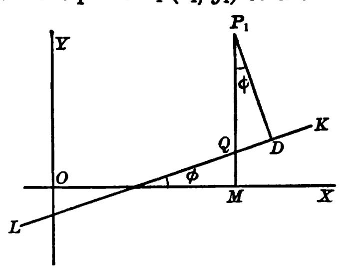

FIG. 26*a*.

From the figure it is seen that

(*b*) $$MP_1 = y_1.$$

Since Q is on the line LK, its coördinates, x_1 and MQ, must satisfy the equation of LK. Therefore

$$Ax_1 + B \cdot MQ + C = 0,$$

and consequently

$$\text{(c)} \qquad MQ = -\frac{Ax_1 + C}{B}.$$

The slope of LK is $\tan\phi = -A/B$, whence

$$\text{(d)} \qquad \cos\phi = \frac{B}{\pm\sqrt{A^2 + B^2}}.$$

Substituting the values from (*b*), (*c*), (*d*) in (*a*),

$$DP_1 = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}. \qquad (26)$$

Equation (26) *gives the distance from the point* (x_1, y_1) *to the line whose equation is* $Ax + By + C = 0$. The distance being positive, such a sign must be used in the denominator that the result is positive.

Example 1. Find the distance from the point (1, 2) to the line $2x - 3y = 6$.

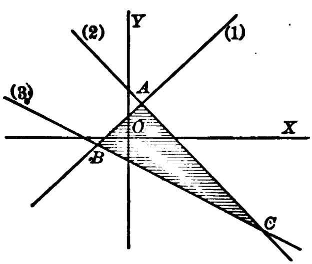

FIG. 26*b*.

The distance from any point (x_1, y_1) to the line is by (26)

$$\frac{2x_1 - 3y_1 - 6}{\pm\sqrt{13}}.$$

The distance from (1, 2) is then

$$\frac{2(1) - 3(2) - 6}{\pm\sqrt{13}} = \frac{10}{\sqrt{13}}.$$

Ex. 2. The lines (1) $y - x - 1 = 0$, (2) $x + y - 2 = 0$, (3) $x + 2y + 2 = 0$ determine a triangle ABC. Find the bisector of the angle A between the lines (1) and (2) (Fig. 26*b*).

The bisector is a locus of points equidistant from the lines (1)

and (2). If (x, y) is any point of the bisector, x and y must then satisfy the equation

$$\frac{y-x-1}{\pm\sqrt{2}} = \frac{x+y-2}{\pm\sqrt{2}}.$$

The signs must be so chosen that these expressions are positive at points inside the triangle. At the origin these expressions become $-1/(\pm\sqrt{2})$, $-2/(\pm\sqrt{2})$. Hence the negative sign must be used in both denominators. The bisector required is therefore

$$\frac{y-x-1}{-\sqrt{2}} = \frac{x+y-2}{-\sqrt{2}}.$$

When simplified this becomes $x = \frac{1}{2}$.

Exercises

Determine the region occupied by points satisfying the inequalities in each of the following cases:

1. $2x + 3y - 6 > 0$.
2. $x < 3y$.
3. $x - y - 1 > 0$,
   $y - 2x > 0$.
4. $2x - y - 2 > 0$,
   $3x + 4y - 12 < 0$,
   $2y - 1 > 0$.
5. $y - 2x > 1$,
   $y - 2x < 3$,
   $2y + x > 1$,
   $2y + x < 3$.
6. $(x + 2y - 3)(2x - y + 3) > 0$.
7. $(x + 4y)^2 > 9$.

8. Inside the triangle determined by the lines $x + y = 0$, $2x - 3y - 1 = 0$, $y - 2 = 0$, what algebraic signs have the expressions $x + y$, $2x - 3y - 1$, $y - 2$?

9. Express by inequalities the inside of the triangle determined by the points (1, 1), (3, 4), (2, −2).

10. Find the distance from the point (3, 5) to the line $y = 4x - 8$.

11. Find the distance from (6, −2) to the line through (−1, 3) and (5, −1).

12. Find the distance between the two parallel lines, $4x+3y-10=0$ and $4x + 3y - 8 = 0$.

13. Find the equation of the bisector of the acute angle between the lines $2x - y = 1$, $x + 3y = 2$.

14. A triangle is formed by the lines $3x - 4y = 5$, $4x + 3y = 5$, $5x + 12y = 13$. Find the center of the inscribed circle.

15. Find the locus of a point whose distance from (2, 3) is equal to its distance from the line $x + 2y = 3$.

16. The sum of the distances from the point $P(x, y)$ to the lines $y = x$, $x + y = 4$, $y + 2 = 0$, is 4. Find an equation satisfied by the coördinates of P. Do all points whose coördinates satisfy this equation have the required property?

Art. 27. Equation of a Circle

A circle is the locus of a point at constant distance from a fixed point. The fixed point is the center of the circle and the constant distance is its radius.

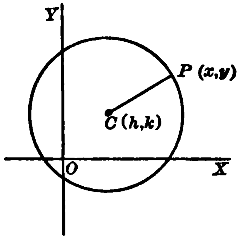

FIG. 27.

Let $C(h, k)$, Fig. 27, be the center of the circle and r its radius. If $P(x, y)$ is any point on the circle

$$r = CP = \sqrt{(x-h)^2 + (y-k)^2},$$

or $(x-h)^2 + (y-k)^2 = r^2$. $\qquad$ (27a)

This is an equation satisfied by the coördinates of any point on the circle. Conversely, if the coördinates x, y satisfy this equation, the point P is at the distance r from the center and consequently lies on the circle. Therefore it is the equation of the circle.

Example 1. Find the equation of the circle with center $(-2, 1)$ and radius 3. In this case, $h = -2$, $k = 1$, $r = 3$.

By (27a) the equation of the circle is then

$$(x+2)^2 + (y-1)^2 = 9.$$

Ex. 2. Find the equation of the circle with center (1, 1) which passes through the point (3, −4).

The radius is the distance from the center to the point (3, −4). Consequently

$$r = \sqrt{(3-1)^2 + (-4-1)^2} = \sqrt{29}.$$

The equation of the circle is therefore

$$(x-1)^2 + (y-1)^2 = 29.$$

Form of the Equation. — Expanding (27a),

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0.$$

This is an equation of the form

$$A(x^2 + y^2) + Bx + Cy + D = 0, \tag{27b}$$

in which A, B, C, D are constant.

Conversely, if A is not zero, an equation of the form (27b) represents a circle, if it represents any curve at all. To show this divide by A and complete the squares of the terms containing x and those containing y separately. The result is

$$\left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \frac{B^2 + C^2 - 4AD}{4A^2}.$$

If the right side of this equation is positive, it represents a circle with center $(-B/2A, -C/2A)$ and radius $\sqrt{B^2 + C^2 - 4AD}/2A$. If the right side is zero, the radius is zero and the circle shrinks to a point. If the right side is negative, since the sum of squares of real numbers is positive, there is no real locus.

Example 1. Find the center and radius of the circle

$$2x^2 + 2y^2 - 3x + 4y = 1.$$

Dividing by 2 and completing the squares,

$$(x - \tfrac{3}{4})^2 + (y + 1)^2 = \tfrac{33}{16}.$$

The center of the circle is $(\frac{3}{4}, -1)$ and its radius is $\frac{1}{4}\sqrt{33}$.

Ex. 2. Determine the locus of

$$x^2 + y^2 + 4x - 6y + 13 = 0.$$

Completing the squares,

$$(x + 2)^2 + (y - 3)^2 = 0.$$

The sum of two squares can only be zero when both are zero. The only real point on this locus is then $(-2, 3)$. The circle shrinks to a point.

Ex. 3. Discuss the locus of

$$x^2 + y^2 + 2y + 3 = 0.$$

Completing the squares,

$$x^2 + (y + 1)^2 = -2.$$

There are no real values for which this is true. The locus is imaginary.

Art. 28. Circle Determined by Three Conditions

When a circle is given, h, k, r, in equation (27a), have definite values. Conversely, if values are assigned to h, k and r, a circle is determined. This is expressed by saying that the circle is determined by three independent constants. Any limitation on the circle, such as requiring it to pass through a point or be tangent to a line, can be expressed by an equation or equations connecting h, k and r. Now three numbers are determined by three independent equations. This is expressed by saying a circle can be found satisfying three independent conditions. Thus a circle can be passed through three points, can touch three lines, etc. Many problems consist in finding a circle doing three things. To find the equation of such a circle, express the three conditions by equations connecting h, k and r, solve and substitute the values in equation (27a).

Example 1. A circle is tangent to the x-axis, passes through (1, 1) and has its center on the line $y = x - 1$. Find its equation.

The circle is shown in Fig. 28a. Since the circle is tangent to the x-axis, $k = \pm r$. Since it passes through (1, 1), the circle lies above the x-axis and k is positive. Hence

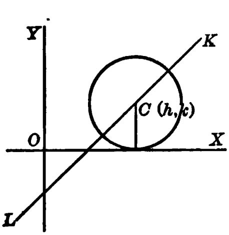

FIG. 28a.

$$(a) \qquad k = r.$$

Since the center is on the line $y = x - 1$, its coördinates must satisfy that equation. Consequently

$$(b) \qquad k = h - 1.$$

Since the circle passes through (1, 1),

$$(c) \qquad (1 - h)^2 + (1 - k)^2 = r^2.$$

The solution of (a), (b) and (c) is $h = 2$, $k = 1$, $r = 1$. The equation required is then

$$(x - 2)^2 + (y - 1)^2 = 1.$$

Ex. 2. Find the equation of the circle through (2, 3), (4, 1) and tangent to the line $4x - 3y = 15$ (Fig. 28b).

Since the circle is tangent to the line its center is at a distance r

from the line. The distance from the center (h, k) to $4x - 3y = 15$ is by equation (26)

$$(a) \qquad \frac{4h - 3k - 15}{\pm 5}.$$

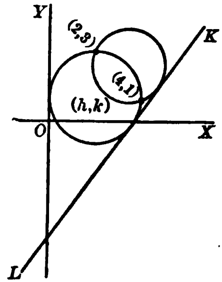

FIG. 28b.

Since the circle passes through (2, 3) its center is on the same side of the line as (2, 3). To make (a) positive at the center and therefore at (2, 3), the negative sign must be used in the denominator of (a). The condition of tangency is then

$$(b) \qquad \frac{4h - 3k - 15}{-5} = r.$$

Since the circle passes through (2, 3) and (4, 1),

$$(c) \qquad (2 - h)^2 + (3 - k)^2 = r^2,$$

$$(d) \qquad (4 - h)^2 + (1 - k)^2 = r^2.$$

Solving equations (b), (c) and (d) simultaneously, we get

$$h = 2, \quad k = 1, \quad r = 2 \text{ and } h = \tfrac{178}{49}, \quad k = \tfrac{129}{49}, \quad r = \tfrac{82}{49}.$$

These are consequently two circles satisfying the given conditions. Their equations are

$$(x - 2)^2 + (y - 1)^2 = 4,$$
$$(x - \tfrac{178}{49})^2 + (y - \tfrac{129}{49})^2 = (\tfrac{82}{49})^2.$$

Ex. 3. Find the circle through the three points (0, 1), (−1, −1) and (2, 0).

The coördinates of the three points must satisfy the equation of the circle. Hence

$$(0 - h)^2 + (1 - k)^2 = r^2,$$
$$(-1 - h)^2 + (-1 - k)^2 = r^2,$$
$$(2 - h)^2 + (0 - k)^2 = r^2.$$

Subtracting the first and third of these equations from the second we get

$$2h + 4k + 1 = 0,$$
$$6h + 2k - 2 = 0.$$

The solution of these equations is $h = \frac{1}{2}$, $k = -\frac{1}{2}$. These values substituted in either of the original equations give $r^2 = \frac{10}{4}$. The circle required is therefore

$$(x - \tfrac{1}{2})^2 + (y + \tfrac{1}{2})^2 = \tfrac{10}{4}.$$

Exercises

Find the center and radius of each of the following circles. Draw the curves.

1. $x^2 + y^2 = 25.$
2. $x^2 + y^2 = 4x.$
3. $2x^2 + 2y^2 - 6y + 1 = 0.$
4. $3x^2 + 3y^2 + 4x = 1.$
5. $x^2 + y^2 = x + y.$
6. $x^2 - 2ax + y^2 - 2ay = 0.$
7. $x^2 + y^2 - 2x + 4y + 2 = 0.$
8. $x^2 + y^2 - 2x + 1 = 0.$

9. What locus is represented by the equation $x^2 + y^2 + 2x + 2y + 2 = 0$?

10. What is the locus of the equation $x^2 + y^2 - 6x + 6y + 9 = 0$?

11. Find the equation of the circle through $(-2, 4)$ and having the same center as the circle $x^2 + y^2 - 5x + 4y - 1 = 0$.

12. Find the equation of the circle whose diameter is the segment joining $(-1, -2)$ and $(3, 4)$.

13. Find the equations of the circles through $(1, 2)$ and tangent to both coördinate axes.

14. Find the equations of the circles with centers at the origin and tangent to the circle $x^2 + y^2 - 4x + 4y + 7 = 0$.

15. Find the intersections of the circles

$$x^2 + y^2 = 2x + 2y,$$
$$x^2 + y^2 + 2x = 4.$$

16. Find the equation of the circle through the three points $(0, 3)$, $(3, 0)$ and $(0, 0)$.

17. Find the circles of radius 5 passing through the points $(2, -1)$ and $(3, -2)$.

18. The center of a circle passing through $(1, -2)$ and $(-2, 2)$ is on the line $8x - 4y + 9 = 0$. What is its equation?

19. A circle passes through the points $(0, 0)$, $(2, -2)$ and is tangent to the line $y + 4 = 0$. Find its equation.

20. A circle passes through the points $(-1, 0)$, $(0, 1)$ and is tangent to the line $x - y = 1$. Find its equation.

21. A circle is tangent to the lines $x = 3$, $x = 7$, and its center is on the line $y = 2x + 4$. What is its equation?

22. Find the equation of the circle circumscribed about the triangle formed by the three lines $x + y - 2 = 0$, $9x + 5y - 2 = 0$, $y + 2x - 1 = 0$.

23. Find the equation of the circle inscribed in the triangle formed by the lines $x + y = 1$, $y - x = 1$, $x - 2y = 1$.

24. Find the locus of points from which the tangents to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 2x + 4y = 4$ are of equal length.

25. By subtracting the equation $x^2 + y^2 - 2x - 2y = 0$ from the equation $x^2 + y^2 + 2x - 6y + 2 = 0$ the equation of a line is obtained. Show that this line is the common chord of the two circles.

26. A point moves so that the sum of its distances from two vertices of an equilateral triangle is equal to its distance from the third vertex. Find an equation satisfied by its coördinates. Do all points whose coordinates satisfy this equation have the required property?

CHAPTER 4

SECOND DEGREE EQUATIONS

Art. 29. The Ellipse

If a circle is deformed in such a way that the distances of its points from a fixed diameter are all changed in the same ratio, the resulting curve is called an ellipse.

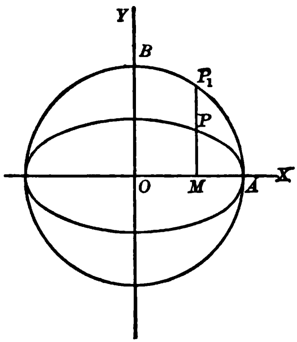

FIG. 29*a*.

For example, in Fig. 29*a*, if each point P_1 of the circle is moved to a point P such that

$$MP/MP_1 = k = \text{constant},$$

the locus of P is an ellipse.

Let the center of the circle be the origin and the fixed diameter the x-axis. Let x_1, y_1 be the coördinates of P_1 and x, y the coördinates of P. Then

$$x_1 = x, \quad y_1 = MP_1 = MP/k = y/k.$$

If a is the radius of the circle, its equation is

$$x_1^2 + y_1^2 = a^2.$$

Replacing x_1 and y_1 by their expressions in terms of x and y,

$$x^2 + \frac{y^2}{k^2} = a^2$$

is found to be the equation of the ellipse. Dividing by a^2 and replacing ka by b, the equation of the ellipse becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{29}$$

When y is zero, x is $\pm a$, and, when x is zero, y is $\pm b$. Hence a and b are the distances, OA and OB, intercepted on the axes.

Since the equation of an ellipse contains only the squares of x and y, to each value of x correspond two values of y differing only in sign. A line perpendicular to the x-axis then cuts the curve in two points P, P' (Fig. 29b) equidistant from the axis. Similarly, a line perpendicular to the y-axis cuts the curve in two points P, P'' equidistant from the y-axis. This is expressed by saying that the curve is symmetrical with respect to both of the coördinate axes. They are called the *axes* of the curve.

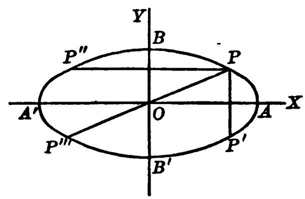

FIG. 29b.

Any line through O cuts the curve in two points $P(x, y)$, $P'''(-x, -y)$ equidistant from O. For this reason the curve is called symmetrical with respect to the origin and the point O is called the *center* of the curve.

The ellipse cuts the axes in four points A', A, B', B. The segments $A'A$ and $B'B$ are sometimes called the axes of the curve. The longer of these is called the *major axis*, the shorter the *minor axis*. The distances OA and OB, equal to a and b, are called the *semi-axes* of the curve. The ends of the major axis are called *vertices*.

Art. 30. The Ellipse in Other Positions

Equation (29) represents an ellipse whose axes are the coördinate axes. The equation can be stated in a form valid in any position. In fact, since $x = NP$, $y = MP$, $a = OA$, $b = OB$, equation (29) is equivalent to

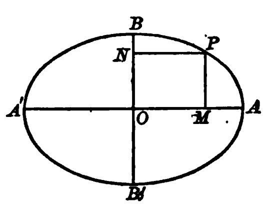

FIG. 30a.

$$\frac{NP^2}{OA^2} + \frac{MP^2}{OB^2} = 1, \qquad (30a)$$

that is, *the ratios, obtained by dividing the squares of the distances of any point on the ellipse from the axes by the squares of the parallel semi-axes, have a sum equal to* 1.

For example, if the center of the ellipse is the point (h, k) and the axes of the curve are parallel to the coördinate axes (Fig. 30b),

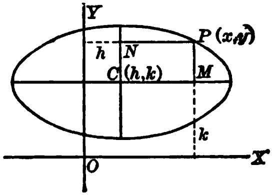

FIG. 30b.

$$NP = x - h, \quad MP = y - k$$

and the equation of the curve is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \qquad (30b)$$

a and b being the semi-axes parallel to OX and OY respectively.

Example 1. Find the equation of the ellipse with vertices A' $(-3, 2)$, A $(5, 2)$ and semi-axes equal to 4 and 1.

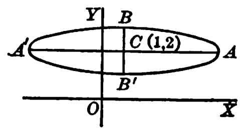

FIG. 30c.

The segment $A'A$ is the major axis (Fig. 30c). Its middle point is the center of the curve. Consequently, the center is C $(1, 2)$. Also the semi-axes are

$$CA = 4, \quad CB = 1.$$

The equation of the ellipse is therefore

$$\frac{(x-1)^2}{16} + \frac{(y-2)^2}{1} = 1.$$

Ex. 2. Show that the equation $9x^2 + 4y^2 + 36x - 24y + 36 = 0$ represents an ellipse. Find its center and axes.

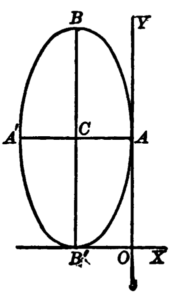

FIG. 30d.

The equation can be written

$$9(x^2 + 4x) + 4(y^2 - 6y) + 36 = 0.$$

Completing the squares in the parentheses,

$$9(x+2)^2 + 4(y-3)^2 = 36,$$

or

$$\frac{(x+2)^2}{4} + \frac{(y-3)^2}{9} = 1.$$

Comparing this with equation (30b) it is seen to represent an ellipse with center $(-2, 3)$. The axes are horizontal and vertical lines through the center. Their equations are $x = -2$, $y = 3$. The

major axis is vertical, the minor axis horizontal. The vertices, at the ends of the major axis, are B' $(-2, 0)$, B $(-2, 6)$.

Ex. 3. Find the equation of the ellipse whose axes are the lines $2x + y = 3$, $x - 2y = 0$, and whose semi-axes along those lines are 1 and 3 respectively.

Let $P(x, y)$ be any point on the curve (Fig. 30e). Then

$$NP = \frac{2x + y - 3}{\pm\sqrt{5}}, \quad MP = \frac{x - 2y}{\pm\sqrt{5}}.$$

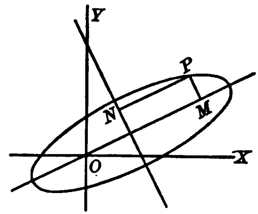

FIG. 30e.

The equation of the ellipse is $NP^2/9 + MP^2/1 = 1$, whence

$$\frac{(2x + y - 3)^2}{45} + \frac{(x - 2y)^2}{5} = 1.$$

Exercises

Make graphs of the following equations. Show that the curves are ellipses. Find their centers and semi-axes.

1. $x^2 + 2y^2 = 6$.
2. $4x^2 + y^2 - 8x + 4y + 7 = 0$.
3. $x^2 + 2y^2 = x + y$.
4. $3x^2 + 2y^2 - 6x + 8y = 1$.
5. $x^2 + 3y^2 + 6x - 6y = 1$.
6. $5x^2 + 2y^2 - 10x + 4y + 7 = 0$.

7. Find the equation of the ellipse with center $(1, -3)$ and semi-axes, parallel to OX and OY, whose lengths are 2 and 3 respectively.

8. Find the equation of the ellipse with center $(-2, 4)$ tangent to both coördinate axes.

9. Find the equation of the ellipse whose axes are the lines

$$x + y - 2 = 0, \quad x - y + 2 = 0,$$

and whose semi-axes along those lines have lengths equal to 1 and 4 respectively.

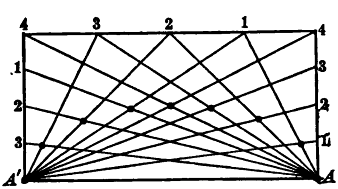

FIG. 30f.

10. Show that an oblique plane section of a right circular cylinder is an ellipse.

11. Three sides of a rectangle are divided into an equal number of parts and the points of division connected by straight lines with the opposite corners as shown in Fig. 30f. Show that the intersections of lines through like numbered

points are on an ellipse with axes equal in length to the sides of the rectangle.

12. Show that the locus of points, the sum of whose distances from two fixed points is constant, is an ellipse. Let the fixed points be $(-c, 0)$, $(+c, 0)$ and let the constant distance be $2a$.

Art. 31. The Parabola

Let LK, RS be perpendicular lines and MP, NP perpendiculars from any point P to them. If a is constant and NP considered positive when P is on one side of RS, negative when on the other, the locus of points P such that

$$MP^2 = a \cdot NP \tag{31a}$$

is called a parabola. *A parabola is thus a locus of points the squares of whose distances from one of two perpendicular lines are proportional to their distances from the other.* The complete locus of such points is two parabolas, one on each side of RS.

To each value of NP correspond two values of MP differing only in sign. The curve is therefore symmetrical with respect to LK

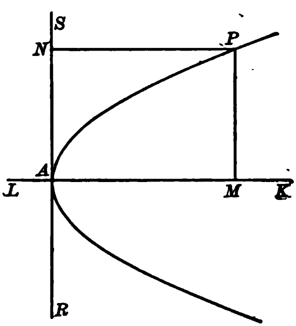

FIG. 31*a*.

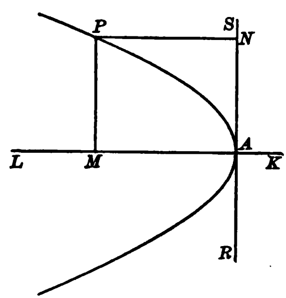

FIG. 31*b*.

which is called the axis of the parabola. The point A is called the vertex. The curve passes through A but, since $a \cdot NP$ is positive, it does not cross RS.

If LK is the x-axis, RS the y-axis, the equation of the curve is $y^2 = ax$. If a is positive x must be positive and the curve is on the

right of the y-axis as in Fig. 31a. If a is negative, x must be negative and the curve is on the left of the y-axis as in Fig. 31b.

If the axis of the parabola is parallel to OX and the vertex is

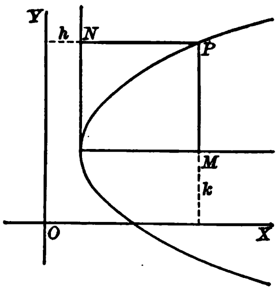

FIG. 31c.

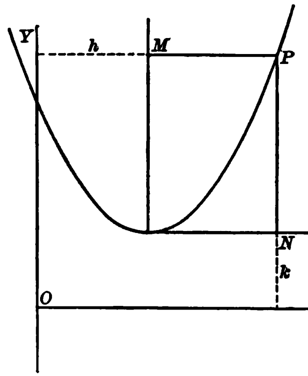

FIG. 31d.

(h, k) (Fig. 31c), $NP = x - h$, $MP = y - k$, and the equation of the parabola is

$$(y - k)^2 = a\,(x - h). \tag{31b}$$

If (h, k) is the vertex and the axis is parallel to OY (Fig. 31d) the equation of the parabola is

$$(x - h)^2 = a\,(y - k). \tag{31c}$$

If the axis of the parabola is not parallel to either coördinate axis, its equation can be obtained from (31a) by expressing MP and NP in terms of the coördinates of P.

Example 1. Show that $y^2 = 3\,x + 2\,y - 4$ is the equation of a parabola. Find its vertex and axis.

Transposing and completing the square, the equation becomes

$$(y - 1)^2 = 3\,x - 3 = 3\,(x - 1).$$

Comparing this with the equation

$$(y - k)^2 = a\,(x - h),$$

it is seen to represent a parabola for which

$$h = 1, \quad k = 1, \quad a = 3.$$

The vertex is the point $(1, 1)$ and the axis is the line $y = 1$.

Ex. 2. Find the equation of the parabola with vertex (1, 2) and axis $y = x + 1$, which passes through the point (3, 7).

The line through the vertex perpendicular to the axis is $x + y - 3 = 0$. If x, y are the coördinates of any point P (Fig. 31*e*), then

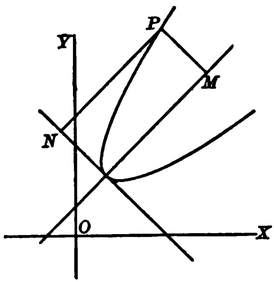

FIG. 31*e*.

$$MP = \frac{y - x - 1}{\pm\sqrt{2}},$$

$$NP = \frac{x + y - 3}{\pm\sqrt{2}}.$$

If P is a point on the parabola, $MP^2 = a \cdot NP$, whence

$$(y - x - 1)^2 = \pm a\sqrt{2}\,(x + y - 3).$$

Since the curve passes through (3, 7)

$$9 = \pm a\sqrt{2}\,(7).$$

This value of a substituted in the previous equation gives

$$7\,(y - x - 1)^2 = 9\,(x + y - 3)$$

as the equation of the parabola.

Ex. 3. An arch has the form of a parabola with vertical axis (Fig. 31*f*). If the arch is 10 feet high at the center and 30 feet wide at its base, find its height at a distance of 5 feet from one end.

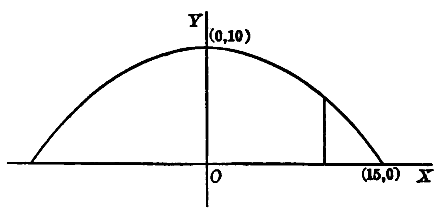

FIG. 31*f*.

Take the origin at the middle point of the base of the arch. The vertex is then (0, 10). The equation of the arch is consequently

$$(x - 0)^2 = a\,(y - 10).$$

The curve crosses the x-axis at (15, 0). Hence

$$15^2 = a\,(-10).$$

Substituting this value of a, the equation of the arch becomes

$$10x^2 = 225(10 - y).$$

At a point 5 feet from one end $x = \pm 10$. The corresponding value of y is 50/9, which is the height of the arch at that point.

Exercises

Make graphs of the following equations. Show that the curves are parabolas. Find their axes and vertices.

1. $y^2 = 8x - 4$.
2. $y^2 = -2x + 1$.
3. $y = x^2 - 2x + 3$.
4. $y = (x - 1)(x + 2)$.
5. $x = y^2 - 3y$.
6. $x^2 - 3x + 2y - 4 = 0$.

7. Find the equation of the parabola with horizontal axis and vertex at the origin, which passes through (3, 4).

8. Find the equation of the parabola with vertical axis and vertex at (−2, 2), which passes through (1, −3).

9. An arch in the form of a parabola with vertical axis is 29 feet across the bottom and its highest point is 8 feet above the base. What is the length of the beam placed horizontally across the arch 4 feet from the top?

10. A cable of a suspension bridge hangs in the form of a parabola with vertical axis. The roadway, which is horizontal and 240 feet long, is supported by vertical wires attached to the cable, the longest being 80 feet and the shortest 30 feet. Find the length of the supporting wire attached to the roadway 40 feet from the middle.

11. A point moves so that its distance from a fixed point is equal to its distance from a fixed line. Show that the locus described is a parabola.

12. Two sides, AB and BC, of a rectangle are divided into an equal number of parts and the points of division numbered as shown in Fig. 31g. Through the points of AB lines are drawn parallel to BC, and through those of BC lines are drawn passing through A. Show that the intersections of lines through like numbered points are on a parabola.

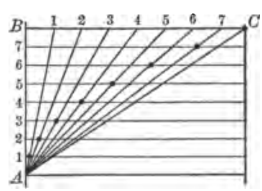

FIG. 31g.

Art. 32. The Hyperbola

Let KL and RS (Fig. 32a) be two straight lines intersecting in C, PM and PN the perpendiculars from any point P to these lines. Let MP be considered positive when P is on one side of KL, nega-

tive when on the other side. Similarly, let NP be positive when P is on one side of RS, negative when on the other side. A hyperbola is the locus of points P such that the product

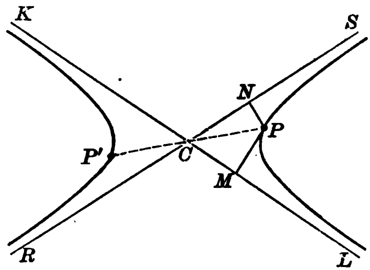

FIG. 32a.

$$MP \cdot NP = \text{constant.} \qquad (32a)$$

If the signs of MP and NP are both changed the product is not changed. Hence if any point P lies on the curve, the point P' at equal distance on the other side of C is on the curve. The hyperbola is thus symmetrical with respect to C which is called the *center*. The curve consists of two parts in a pair of vertical angles determined by KL and RS. *The hyperbola is a locus of points the product of whose distances from two lines is constant.* The complete locus of such points is however two hyperbolas, one in each pair of vertical angles between the lines.

If MP is very large, NP must be very small and conversely. As it goes to an indefinite distance the curve thus approaches indefinitely near the lines KL and RS. They are called *asymptotes*.

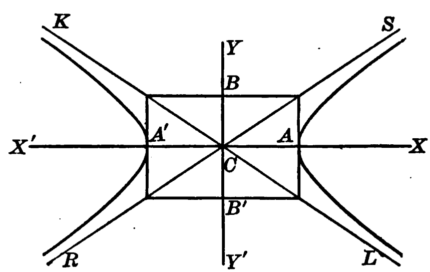

FIG. 32b.

Let C be the origin (Fig. 32b) and take as x-axis the line bisecting the vertical angles in which the curve lies. The curve crosses the x-axis at two points A', A. Construct the rectangle with sides through A' and A, having KL and RS as diagonals. Let $CA = a$, $CB = b$. The equations of KL and RS are then

$$y = \pm (b/a)\, x. \qquad (32b)$$

Consequently, Fig. 32a,

$$MP = \frac{bx + ay}{\pm\sqrt{b^2 + a^2}}, \qquad NP = \frac{bx - ay}{\pm\sqrt{b^2 + a^2}}.$$

If P is a point on the hyperbola, $MP \cdot NP =$ const., whence

$$b^2x^2 - a^2y^2 = \pm (b^2 + a^2) \text{ const.} = k.$$

Since the hyperbola passes through A (a, o)

$$b^2a^2 = k.$$

Substituting this value of k and dividing by a^2b^2, the equation of the hyperbola becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{32c}$$

Since the equation contains only squares of x and y the curve is symmetrical with respect to the coördinate axes. They are called the *axes* of the curve. The axis cutting the curve is called *transverse*, the one not cutting the curve is called *conjugate*. The distances $CA = a$, $CB = b$ are called the *semi-axes*. The points A' and A, where the transverse axis cuts the curve, are called *vertices*.

Equation (32c) represents the hyperbola referred to its axes, the x-axis being transverse. If the transverse axis is parallel to the x-axis and the center is (h, k) (Fig. 32c),

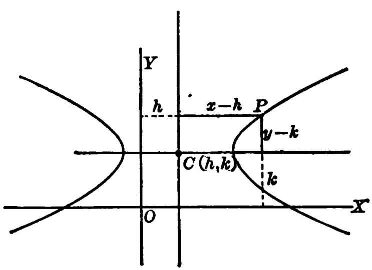

FIG. 32c.

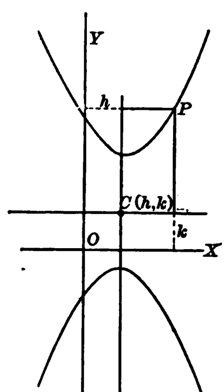

FIG. 32d.

the coördinates relative to the axes are $x - h$ and $y - k$. The

equation of the hyperbola is then

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1. \tag{32d}$$

If the transverse axis is parallel to OY (Fig. 32d) and the center is (h, k) the equation of the hyperbola is

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1. \tag{32e}$$

If the axes of the curve are not parallel to the coördinate axes, its equation can be obtained by using the definition or by replacing x and y in equation (32c) by the distances of a point P from the axes of the curve.

Art. 33. The Rectangular Hyperbola

If the asymptotes of a hyperbola are perpendicular to each other it is called rectangular. In this case the asymptotes are usually

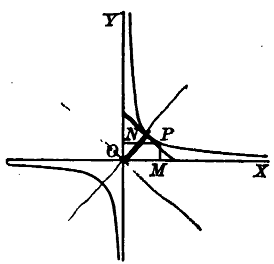

FIG. 33a.

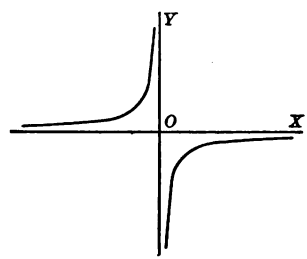

FIG. 33b.

taken as coördinate axes. The definition, $MP \cdot NP =$ const., gives the equation of the curve in the form

$$xy = k. \tag{33a}$$

If k is positive the curve lies in the first and third quadrants (Fig. 33a), if k is negative it lies in the second and fourth quadrants (Fig. 33b).

The axes of the rectangular hyperbola make angles of 45° with the asymptotes. The rectangle, Fig. 32b, is a square and $a = b$. If

the axes of the rectangular hyperbola are taken as axes of coördinates, its equation is then

$$x^2 - y^2 = a^2, \tag{33b}$$

the x-axis being transverse.

Example 1. Show that $9x^2 - 4y^2 + 18x + 16y - 43 = 0$ is the equation of a hyperbola. Find its center, axes and asymptotes.

The equation can be written

$$9(x^2 + 2x) - 4(y^2 - 4y) = 43.$$

Completing the squares,

$$9(x+1)^2 - 4(y-2)^2 = 36,$$

or

$$\frac{(x+1)^2}{4} - \frac{(y-2)^2}{9} = 1.$$

Comparing this with equation (32*d*), it is seen to represent a hyperbola with center $(-1, 2)$ and semi-axes, $a = 2$, $b = 3$. The transverse axis is the line $y = 2$. The conjugate axis is $x = -1$. The asymptotes are lines through the center with slopes $\pm b/a$. Their equations are consequently

$$y - 2 = \pm \tfrac{3}{2}(x + 1).$$

Ex. 2. Show that $xy = 2x - 3y$ is the equation of a rectangular hyperbola. Find its center, axes and asymptotes.

The equation can be written

$$(x+3)(y-2) = -6.$$

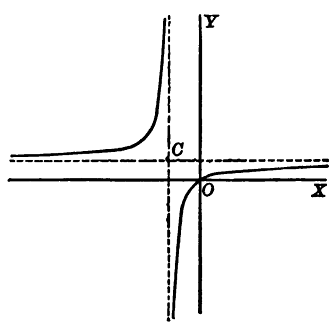

FIG. 33c.

The quantities $x + 3$ and $y - 2$ are coördinates of $P(x, y)$ with respect to axes through $(-3, 2)$ parallel to OX and OY (Fig. 33*c*). The curve is therefore a rectangular hyperbola with center $(-3, 2)$ and asymptotes $x = -3$ and $y = 2$. The axes pass through $(-3, 2)$ making angles of 45°

with the asymptotes. Their equations are consequently

$$y - 2 = \pm(x + 3).$$

Ex. 3. Find the equation of the hyperbola with asymptotes $x - y = 1$ and $x = 2$, which passes through (3, 4).

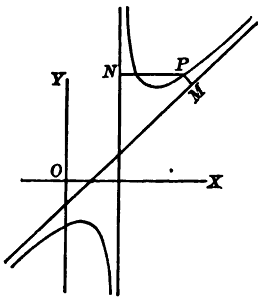

FIG. 33*d*.

Let $P\ (x, y)$ be a point on the curve. Then (Fig. 33*d*)

$$MP = \frac{x - y - 1}{\pm\sqrt{2}}, \quad NP = x - 2.$$

The equation of the curve is $MP \cdot NP$ = constant. Consequently,

$$(x - y - 1)(x - 2) = \text{constant}.$$

Since the curve passes through (3, 4), the constant in this equation is $(3 - 4 - 1)(3 - 2) = -2$. The equation required is then

$$(x - y - 1)(x - 2) = -2.$$

Ex. 4. Find the equation of the hyperbola with center at the origin, transverse axis $y - 2x = 0$, which passes through (0, 2) and has the x-axis as an asymptote.

The conjugate axis, being perpendicular to the transverse axis at the center, is $x + 2y = 0$. In equation (32*c*) x and y can be replaced by the distances of a point on the hyperbola from the axes of the curve. In the present case these distances are $(x + 2y)/\sqrt{5}$ and $(y - 2x)/\sqrt{5}$. Hence the equation of the curve is

$$\frac{(x + 2y)^2}{5a^2} - \frac{(y - 2x)^2}{5b^2} = 1.$$

Since the curve passes through (0, 2) $16/5a^2 - 4/5b^2 = 1$. Since the x-axis is an asymptote, there must be no point of intersection of the curve and x-axis. When y is zero the equation becomes $x^2(1/5a^2 - 4/5b^2) = 1$. This will fail to determine a value of x if $1/5a^2 - 4/5b^2 = 0$. This and the previous equation solved simultaneously give $a^2 = 3$, $b^2 = 12$. The equation required is

therefore

$$\frac{(x+2y)^2}{15} - \frac{(y-2x)^2}{60} = 1.$$

Exercises

Show that the following equations represent hyperbolas. Find their centers, axes and asymptotes.

1. $4x^2 - 3y^2 + 12y = 24$.
2. $5x^2 - y^2 - 2y = 4$.
3. $xy + x - y = 3$.
4. $2x^2 - 3y^2 + 4x + 12y = 4$.
5. $x^2 - y^2 - 2x - 6y = 8$.
6. $2xy = 3x - 4$.
7. Find the equations of the two hyperbolas with center $(2, -1)$ and semi-axes, parallel to OX and OY, whose lengths are 1 and 4 respectively.
8. Find the equation of the hyperbola with center $(-2, 1)$, and axes parallel to the coördinate axes, passing through $(0, 2)$ and $(1, -4)$.
9. Find the equations of the hyperbolas whose axes are the lines $3x + 2y = 0$, $2x - 3y = 0$ and whose semi-axes along those lines are equal to 2 and 5 respectively.
10. Show that the locus of a point, the difference of whose distances from two fixed points is constant, is a hyperbola. Let the fixed points be $(-c, 0)$, $(+c, 0)$ and let the difference of the distances be $2a$.
11. A point moves so that the product of the slopes of the lines joining it to $(-a, 0)$ and $(a, 0)$ is constant. Show that it describes an ellipse or a hyperbola.

Art. 34. The Second Degree Equation

An equation of the second degree in rectangular coördinates has the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{34a}$$

A, B, C, D, E, F being constants. The equations of the circle, ellipse, parabola and hyperbola are all of this kind. If the polynomial forming the left side of the equation can be resolved into a product of first degree factors, the equation is said to be *reducible*. If the polynomial cannot be so factored the equation is called *irreducible*.

Reducible Equations. — If the polynomial forming the left side of (34a) can be resolved into a product of first degree factors, the equation has the form

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0.$$

Since a product is zero when and only when one of its factors is zero, the equation is satisfied by all values of x and y such that either

$$a_1x + b_1y + c_1 = 0$$

or

$$a_2x + b_2y + c_2 = 0$$

and by no others. If the coefficients a_1, b_1, etc., are all real these equations represent straight lines. The locus of the equation is then a pair of straight lines (a single line if the factors are equal). If some of the coefficients are imaginary, the locus will usually be a single point whose coördinates make both factors vanish.

Example 1. Determine the locus of the equation $2x^2 - xy - 3y^2 = 0$.

The equation is equivalent to

$$(x + y)(2x - 3y) = 0.$$

The locus is two lines, $x + y = 0$ and $2x - 3y = 0$, passing through the origin.

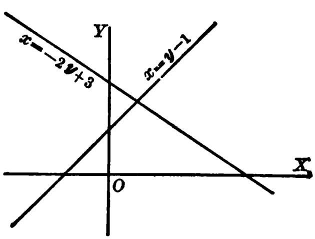

FIG. 34a.

Ex. 2. Determine the locus represented by the equation

$$x^2 + xy - 2y^2 - 2x + 5y - 3 = 0.$$

Arranged in powers of x the equation is

$$x^2 + (y-2)x - (2y^2 - 5y + 3) = 0.$$

Solving by the quadratic formula,

$$\begin{aligned} x &= -\tfrac{1}{2}(y-2) \pm \tfrac{1}{2}\sqrt{9y^2 - 24y + 16} \\ &= -\tfrac{1}{2}(y-2) \pm \tfrac{1}{2}(3y - 4). \end{aligned}$$

There are then two solutions $x = y - 1$ and $x = -2y + 3$. The original equation is satisfied if either of these is satisfied. The locus is two straight lines.

Ex. 3. Determine the locus of the equation

$$x^2 + 3y^2 - 2x + 12y + 13 = 0.$$

When the squares are completed this becomes

$$(x - 1)^2 + 3(y + 2)^2 = 0.$$

The sum of squares of real numbers can only be zero when all are zero. The only real numbers satisfying this equation are then

$$x = 1, \quad y = -2.$$

The polynomial has imaginary factors, $(x - 1) \pm (y - 2)\sqrt{-3}$, but the locus has one real point.

Irreducible Equations. — It will be shown later (Art. 59) that the locus of an irreducible equation of the second degree is an ellipse, parabola, hyperbola or entirely imaginary. A circle is considered as an ellipse with axes of equal length.

By the second degree part of an equation of the second degree is meant the part

$$Ax^2 + Bxy + Cy^2 \tag{34b}$$

containing the terms of second degree. We shall now show how to determine by inspection of this second degree part whether a given second degree equation represents an ellipse, parabola or hyperbola.

An ellipse whose axes are the lines

$$A_1x + B_1y + C_1 = 0, \quad A_2x + B_2y + C_2 = 0$$

is represented by the equation

$$\frac{1}{a^2}\left(\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}}\right)^2 + \frac{1}{b^2}\left(\frac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2}}\right)^2 = 1.$$

The second degree part of this equation is

$$\frac{(A_1x + B_1y)^2}{a^2(A_1^2 + B_1^2)} + \frac{(A_2x + B_2y)^2}{b^2(A_2^2 + B_2^2)}. \tag{1}$$

Since this is a sum of squares it has imaginary factors.

If the axis of a parabola is $A_1x + B_1y + C_1 = 0$, and the line through the vertex perpendicular to the axis is $A_2x + B_2y + C_2 = 0$, the equation of the curve is

$$\left(\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}}\right)^2 = a\left(\frac{A_2x + B_2y + C_2}{\pm\sqrt{A_2^2 + B_2^2}}\right).$$

The second degree part of this equation is

$$\frac{(A_1x + B_1y)^2}{A_1^2 + B_1^2}, \tag{2}$$

which is a complete square.

If the asymptotes of a hyperbola are

$$A_1x + B_1y + C_1 = 0, \quad A_2x + B_2y + C_2 = 0,$$

its equation is

$$\left(\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}}\right)\left(\frac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2}}\right) = \text{constant.}$$

The second degree part of the equation is

$$\left(\frac{A_1x + B_1y}{\sqrt{A_1^2 + B_1^2}}\right)\left(\frac{A_2x + B_2y}{\sqrt{A_2^2 + B_2^2}}\right), \tag{3}$$

which is a product of real first degree factors.

Inspection of (1), (2) and (3) shows that the equations of ellipse, parabola and hyperbola are distinguished by the fact that *the second degree part has imaginary factors in case of the ellipse, is a complete square in case of the parabola, and has real and distinct factors in case of the hyperbola.*

Example 1. Show that $8x^2 - 8xy + 2y^2 = 2x - 3$ is the equation of a parabola.

Solving for y,

$$y = 2x \pm \tfrac{1}{2}\sqrt{4x - 6}.$$

Since y is an irrational function of x the equation is irreducible. That the curve is real is shown by plotting (Fig. 34*b*). The second degree part of the equation is

$$8x^2 - 8xy + 2y^2 = 2(2x - y)^2.$$

This being a square the curve is a parabola.

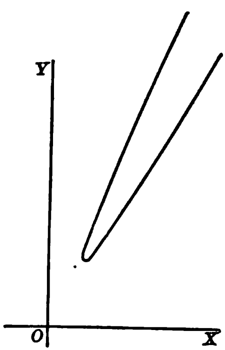

FIG. 34*b*.

Ex. 2. Show that $x^2 + xy + y^2 = 3$ is the equation of an ellipse.

Solving for y,

$$y = \tfrac{1}{2}(-x \pm \sqrt{12 - 3x^2}).$$

The equation is irreducible and represents a real curve. The second degree part is

$$x^2 + xy + y^2 = (x + \tfrac{1}{2}y)^2 + \tfrac{3}{4}y^2.$$

This, being a sum of squares, has imaginary factors and the curve is an ellipse (Fig. 34c).

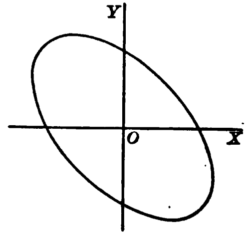

FIG. 34c.

Ex. 3. Show that $x^2 + xy = 2$ is the equation of a hyperbola.

The equation is irreducible and represents a real curve. The second degree part of the equation

$$x^2 + xy = x(x + y)$$

has real and distinct factors. The curve is therefore a hyperbola

Exercises

Plot and determine the nature of the loci represented by the following equations:

1. $x^2 + 2xy - 3y^2 = 0.$
2. $2x^2 + 3xy - y^2 = 0.$
3. $4x^2 + 12xy + 9y^2 = 0.$
4. $3x^2 - 2xy + 3y^2 = 0.$
5. $2x^2 - xy - y^2 - 4x + y + 2 = 0.$
6. $x^2 + 2xy + y^2 + 4x + 4y + 4 = 0.$
7. $xy = 7x.$
8. $x^2 - 2xy + y^2 = 4x.$
9. $x^2 + y^2 = 2x - 3y + 4.$
10. $x^2 + 2xy + 2y^2 = 5.$
11. $x^2 - xy = 4y.$
12. $4x^2 - 4xy + y^2 = 4x - 5y.$
13. $5x^2 + 6xy + 2y^2 - 4x - 2y + 1 = 0.$
14. $(x - y + 3)(2x - y - 1) = 4.$
15. $xy = 2x + 2y - 4.$
16. $x^2 - 2\sqrt{2}xy + 2y^2 = 4x.$
17. $2x^2 + 2y^2 - 4x + 6y = 7.$
18. $3x^2 + 2xy + 2y^2 - 4x - 4y = 4.$
19. $4x^2 + 3xy - 2y^2 + 4x + 7y - 6 = 0.$
20. $xy = 3y - 2x + 6.$

Art. 35. Locus Problems

A locus is often defined by a property of a moving point. The locus is the totality of points having the property. A pair of coördinate axes being given, the equation of the locus is an equation satisfied by the coördinates of every point on it and by no others. To find this equation choose as axes whatever perpendicular lines seem most convenient and let (x, y) be any point of the locus. In terms of x, y and any constant quantities occurring in the problem, express the property used as definition of the locus. The result will be an equation of the locus. In some cases this result can be reduced to a simpler form.

Example 1. The vertices A and B of a triangle ABC are fixed (Fig. 35*a*). Find the locus of the vertex C if $A + B = \frac{3}{4}\pi$.

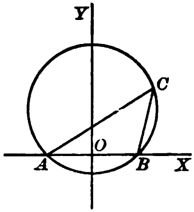

FIG. 35*a*.

The angle C will be

$$C = \pi - A - B = \frac{\pi}{4}.$$

Since this angle is constant the locus is a circle. To find its equation take the middle point of AB as origin and the line AB as x-axis. Let $AO = OB = a$. Then A and B are $(-a, 0)$ and $(a, 0)$. The definition of the locus is $A + B = \frac{3}{4}\pi$, whence

$$\tan(A + B) = -1 = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Now $\tan A$ is the slope of AC and $\tan B$ is the negative of the slope of BC. Hence

$$\tan A = \frac{y}{x + a}, \qquad \tan B = \frac{-y}{x - a}.$$

Substituting these values in the expression for $\tan(A + B)$, the equation of the circle is found to be

$$x^2 + y^2 = 2\,ay + a^2.$$

Ex. 2. A segment has its ends in the coördinate axes and determines with them a triangle of constant area. Find the locus of the middle point of the segment.

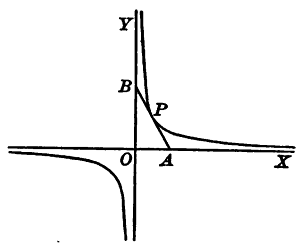

FIG. 35*b*.

Let the segment be AB (Fig. 35*b*). Let $OA = a$, $OB = b$. The area of the triangle OAB is

$$K = \tfrac{1}{2}ab,$$

K being constant. If x and y are the coördinates of the middle point P, then $a = 2\,x$, $b = 2\,y$ and

$$K = 2\,xy.$$

This is an equation satisfied by the coördinates of any point on the locus. Conversely, if the coördinates of any point satisfy this

equation, the segment AB whose intercepts are $OA = 2x$, $OB = 2y$ will have P as its middle point and will determine with the coördinate axes a triangle of area K. Therefore $K = 2xy$ is the equation of the locus. The curve is a rectangular hyperbola with the axes as asymptotes.

Exercises

1. A point moves so that the sum of the squares of its distances from the four sides of a square is equal to twice the area of the square. Find its locus.

2. A point moves so that its shortest distance from a fixed circle is equal to its distance from a fixed diameter of the circle. Find its locus.

3. In a triangle ABC, A and B are fixed. Find the locus of C, if $A - B = \frac{1}{2}\pi$.

4. A point moves so that the sum of the squares of its distances from the three sides of an equilateral triangle is equal to the square of one side of the triangle. Find its locus.

5. A point moves so that the square of its distance from the base of an isosceles triangle is equal to the product of its distances from the other two sides. What is its locus? Show that it passes through the vertices of the two base angles.

6. On a level plane the crack of a rifle and the thud of the bullet striking the target are heard at the same instant. Find the locus of the hearer.

7. A point moves so that the ratio of its distance from a fixed point to its distance from a fixed straight line is a constant e. Show that the locus is an ellipse if $e < 1$, a parabola if $e = 1$ and a hyperbola if $e > 1$.

8. AB and CD are two segments bisecting each other at right angles. Show that the locus of a point P which moves so that $PA \cdot PB = PC \cdot PD$ is a rectangular hyperbola.

9. OA and OB are fixed straight lines, P any point, and PM, PN the perpendiculars from P on OA, OB. Find the locus of P if the quadrilateral $OMPN$ has a constant area.

10. AB is a fixed diameter of a circle and AC is any chord; P and Q are two points on the line AC such that $QC = CP = CB$. Find the locus of P and Q as AC turns about A.

CHAPTER 5

GRAPHS AND EMPIRICAL EQUATIONS

The equation of a curve being given, any number of points on the curve can be constructed by assigning values to either coördinate, calculating the corresponding values of the other coördinate and plotting the resulting points. When enough points have been located a smooth curve drawn through them may be taken as an approximation to the required curve. We desire as quickly as possible to obtain a satisfactory approximation. To some extent this is accomplished by plotting points rather sparsely where the curve is nearly straight and more closely where it bends rapidly. The following are some of the things it may be helpful to note:

(1) Points where the curve crosses the axes and the side of an axis on which it lies between two consecutive crossings (Art. 36).

(2) Values of either coördinate for which the other coördinate is real and values for which it is imaginary (Art. 37).

(3) Symmetry (Art. 38).

(4) Infinite values of the coördinates. Asymptotes (Art. 39).

(5) Direction of the curve near a point (Art. 40).

Art. 36. Intersections with the Axes

The points where a curve meets the x-axis are found by letting $y = 0$ in the equation and solving for x. Similarly, points on the y-axis are found by letting $x = 0$ and solving for y. The x-coördinates of the points on the x-axis and the y-coördinates of the points on the y-axis are called the *intercepts* of the curve on the coördinate axes.

Example 1. $y = x(x - 1)(x + 2)$. The curve crosses the x-axis where $y = 0$, that is, where $x = 0, 1, -2$. These points divide the x-axis and the curve into four parts; namely, the part on the

left of $x = -2$, that between $x = -2$ and $x = 0$, that between $x = 0$ and $x = 1$ and the part on the right of $x = 1$. Construct these parts separately.

On the left of $x = -2$, each factor of $x(x-1)(x+2)$ is negative and consequently the whole product is negative. Hence on the left of $x = -2$, y is negative and the curve lies below the x-axis. Between $x = -2$ and $x = 0$, x and $x - 1$ are negative and $x + 2$ is positive. The product is then positive and so the curve lies above the x-axis. Similarly between $x = 0$ and $x = 1$ the curve is below the x-axis and on the right of $x = 1$ it is above. Using these facts and plotting a few points on each part of the curve the graph of Fig. 36*a* is obtained.

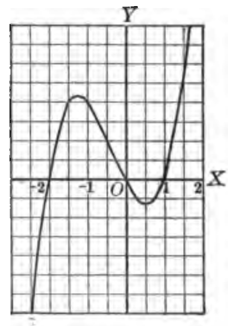

FIG. 36*a*.

Ex. 2. $y - 2 = x^2(x+2)^2$. The curve meets the line $y = 2$ at $x = 0, -2$. Since $x^2(x+2)^2$ is never negative, y can never be less than 2. Hence the curve touches but does not cross the line $y = 2$ at $x = 0$ and $x = -2$. (Fig. 36*b*.)

Ex. 3. $x = y^4 + 2y^3 + 3y^2 + 4y + 2$. The curve meets the y-axis where $x = 0$, that is, where $y^4 + 2y^3 + 3y^2 + 4y + 2 = 0$. Proceeding as

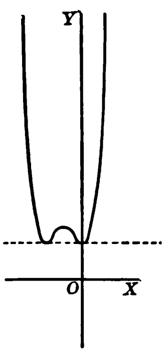

FIG. 36*b*.

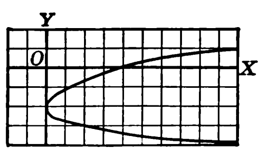

FIG. 36*c*.

in Art. 4 it is found that -1 is a root of this equation. Hence $y + 1$ is a factor of the polynomial. Division gives

$$x = (y+1)(y^3 + y^2 + 2y + 2).$$

In the same way it is found that

$$y^3 + y^2 + 2y + 2 = (y + 1)(y^2 + 2).$$

Consequently,

$$x = (y + 1)^2 (y^2 + 2).$$

Since $y^2 + 2 = 0$ has no real roots, the curve meets the y-axis only at $y = -1$. The factor $(y + 1)^2$ is positive both above and below $y = -1$. Hence x is always positive and the curve does not cross the y-axis. After plotting a few points the curve in Fig. 36c is obtained.

Art. 37. Real and Imaginary Coördinates

When the equation of the curve is of even degree in one of the coördinates, that coördinate may be real for certain values of the other coördinate and imaginary for certain others. The plane is then divided into strips (Fig. 37a) containing a part of the curve and strips not containing a part. These strips being determined the part of the curve in each strip is plotted separately.

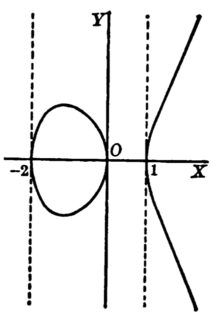

FIG. 37a.

Example 1. $y^2 = x(x - 1)(x + 2)$. The curve crosses the x-axis at $x = -2$, $x = 0$ and $x = 1$. The lines $x = -2$, $x = 0$ and $x = 1$ divide the plane into four parts. On the left of $x = -2$, the product $x(x - 1)(x + 2)$ is negative and y is imaginary. Between $x = -2$ and $x = 0$ the product is positive and y is real. Similarly, between $x = 0$ and $x = 1$, y is imaginary and, on the right of $x = 1$, y is again real. The curve therefore consists of two pieces, one between $x = -2$ and $x = 0$ and the other on the right of $x = 1$. The equation can be written

$$y = \pm\sqrt{x(x - 1)(x + 2)}.$$

To each value of x correspond two values of y differing only in sign. The curve therefore consists of points at equal distances above and below the x-axis (Fig. 37a).

Ex. 2. $x^2 - 4xy + 4y^2 - y - 1 = 0$. Solving for x,

$$x = 2y \pm \sqrt{y+1}.$$

The value of x is real if $y > -1$, imaginary if $y < -1$. The curve therefore lies above the line $y = -1$. It consists of pairs of points at equal distances right and left of the line $x = 2y$ (Fig. 37*b*).

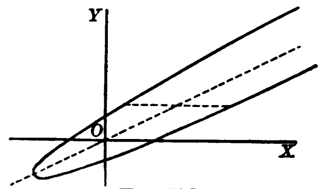

Fig. 37*b*.

Art. 38. Symmetry

Two points P, P' are said to be *symmetrical with respect to a line* if the segment PP' is bisected perpendicularly by that line. In Fig. 38*a*, P and P' are symmetrical with respect to the x-axis, P and P'', Q and R with respect to the y-axis.

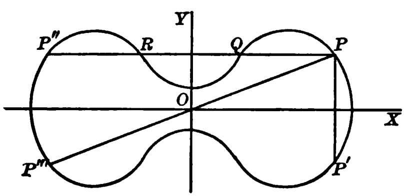

Fig. 38*a*.

Two points P and P''' are called *symmetrical with respect to a point O* if the segment PP''' is bisected by O.

A curve is called *symmetrical with respect to an axis* if all chords perpendicular to the axis meet the curve in pairs of points symmetrical with respect to the axis. The curve in Fig. 38*a* is symmetrical with respect to both coördinate axes.

A curve is called *symmetrical with respect to a center* if all chords through the center meet the curve in pairs of points symmetrical with respect to the center. The curve in Fig. 38*a* is symmetrical with respect to the origin.

A curve $f(x, y) = 0$ is symmetrical with respect to the x-axis if

$$f(x, y) = f(x, -y).$$

For then any point $P(x_1, y_1)$ being on the curve the point $P'(x_1, -y_1)$ is also on the curve. Hence any line $x = x_1$ perpendicular to the x-axis and meeting the curve in a point P will meet it in two points P, P' symmetrical with respect to the x-axis. In particular, *a curve is symmetrical with respect to the x-axis if its equation contains*

only even powers of y. Similarly, the curve $f(x, y) = 0$ is symmetrical with respect to the y-axis if

$$f(x, y) = f(-x, y),$$

and, in particular, *a curve is symmetrical with respect to the y-axis if its equation contains only even powers of x.*

The curve $f(x, y) = 0$ is symmetrical with respect to the origin if

$$f(x, y) = \pm f(-x, -y).$$

For then if $P(x_1, y_1)$ is on the curve $P'''(-x_1, -y_1)$ is also on the curve, and so any line through the origin meeting the curve in a point P will meet it in two points P, P''' symmetrical with respect to the origin. In particular, *a curve is symmetrical with respect to the origin if all the terms in its equation are of even degree or if all are of odd degree.*

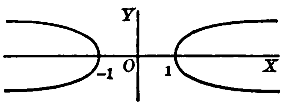

FIG. 38b.

Example 1. $y^2 = \dfrac{x^2 - 1}{x^2 + 1}$. The curve crosses the x-axis at $x = \pm 1$. The value of y is real only when x is in absolute value equal to or greater than 1. There are consequently two parts of the curve, one on the right of $x = 1$, the other on the left of $x = -1$. Since the equation contains only even powers of x and y, the curve is symmetrical with respect to both coördinate axes and with respect to the origin. Since $x^2 - 1 < x^2 + 1$, y is always less than 1. When x is very large, however, y is nearly 1.

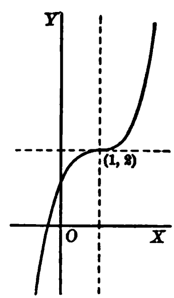

FIG. 38c.

Ex. 2. $y = x^3 - 3x^2 + 3x + 1$. This equation can be written

$$y - 2 = (x - 1)^3.$$

The expressions $y - 2$ and $x - 1$ are the coördinates of a point $P(x, y)$ relative to the lines $y = 2$, $x = 1$ used as axes. Since the equation contains only odd powers of $y - 2$ and $x - 1$ the curve is symmetrical with respect to the point (1, 2).

Art. 39. Infinite Values

In some cases a variable increases in absolute value beyond any assignable bound. Such a variable is said to become infinite and a fictitious value represented by the symbol ∞ is attached to it. But the symbol and the name infinity are used only to express that a variable goes beyond all bounds.

Zero and infinity have the following relations:

$$(1) \quad \frac{0}{a} = 0, \quad \frac{a}{0} = \infty, \quad a \cdot \infty = \infty \cdot a = \infty, \text{ if } a \text{ is not zero.}$$

$$(2) \quad \frac{\infty}{a} = \infty, \quad \frac{a}{\infty} = 0, \quad a \cdot 0 = 0 \cdot a = 0, \text{ if } a \text{ is not infinite.}$$

No definite value can be assigned to the symbols $0/0$, ∞/∞, $0 \cdot \infty$ and $\infty - \infty$.

These relations become evident when 0 and ∞ are interpreted as less than any assignable quantity and greater than any assignable quantity respectively.

A branch of a curve extending to an infinite distance can only be traced until it runs off the diagram. In such cases the curve usually consists of two or more pieces not connected together. Sometimes two pieces are related as at A, B, Fig. 39a, the one going off in a certain direction, the other returning from that direction. Sometimes they are related as at C, D, the one going off one side of the paper, the other returning from the other side. Sometimes there is no return branch.

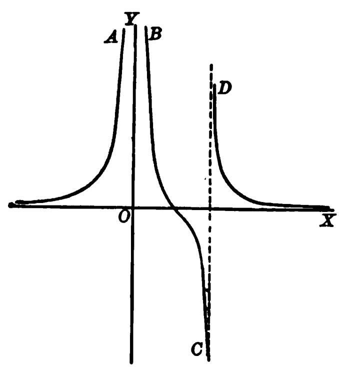

FIG. 39a.

If a branch of a curve when indefinitely prolonged approaches a straight line in such a way that the distance between the two approaches zero, the straight line is called an *asymptote* of the

curve. Both coördinate axes and the line CD are asymptotes of the curve, Fig. 39a. If when indefinitely prolonged the distance between a branch of one curve and a branch of another approaches zero, the two curves are called *asymptotic.*

Example 1. $y = \dfrac{x-1}{x^2(x-2)}$. The graph is shown in Fig. 39a. It crosses the x-axis at $x = 1$. The value of y is infinite when $x = 0$ or 2. When x is a little less than zero, $x - 1$ and $x - 2$ are negative and so y is positive. When x is a little greater than zero y is again positive. In both cases y is very large. The curve thus goes up one side of the y-axis and comes down the other. When x is a little less than 2, y is negative but when x is a little greater than 2, y is positive. The curve then goes down the left side of CD and reappears at the top. As x increases indefinitely, y approaches zero. Consequently, at a great distance from the origin the curve comes closer and closer to the x-axis. The two axes and the line CD are asymptotes of the curve.

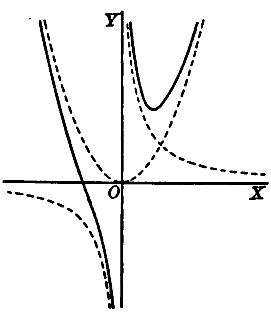

Fig. 39b.

Ex. 2. $y = x^2 + \dfrac{1}{x}$. When x is very small, x^2 is very small and y is approximately $1/x$. Near the y-axis the curve is then asymptotic to the hyperbola $y = 1/x$. The y-axis is an asymptote to both curves. When x is very large, $1/x$ is very small and y is approximately equal to x^2. As x increases indefinitely, the curve therefore approaches the parabola $y = x^2$ to which it is consequently asymptotic.

Art. 40. Direction of the Curve

To determine the shape of a curve near a particular point it is often useful to find the direction along which the curve or a branch of the curve approaches that point. In Fig. 40a, for example, are shown three ways that a branch of a curve can approach a hori-

zontal line. In (1) the curve and line are tangent, in (2) the curve and line intersect at an acute angle and in (3) they are perpendicular

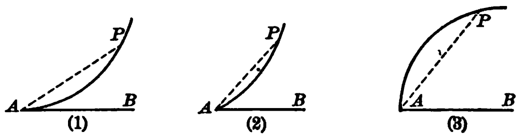

FIG. 40a.

to each other. Let P be a variable point on the curve. As P approaches A the slope of the line AP will approach zero in (1), a finite value not zero in (2) and an infinite value in (3). By finding this slope and determining its limit the direction of the curve near A can be determined.

Example 1. $y^2 = x^3$. The curve passes through the origin (Fig. 40*b*). If x is negative, y is imaginary. The curve therefore reaches the y-axis at the origin but does not cross it. Since the equation contains only even powers of y, the curve is symmetrical with respect to the x-axis. The slope of OP is

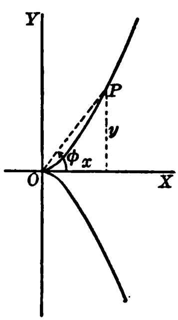

FIG. 40*b*.

$$\tan\phi = y/x.$$

From the equation of the curve, $y = \pm x^{\frac{3}{2}}$. Hence

$$\tan\phi = \pm x^{\frac{3}{2}}/x = \pm x^{\frac{1}{2}}.$$

As P approaches the origin, x approaches zero and consequently $\tan\phi$ approaches zero. The branches of the curve above and below the x-axis are thus tangent to the x-axis and to each other at the origin.

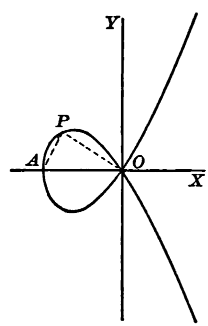

FIG. 40*c*.

Ex. 2. $y^2 = x^2(x + 2)$. The curve crosses the x-axis at the origin and at A $(-2, 0)$ (Fig. 40*c*). There is no point of the curve to the left of $x = -2$. Let P (x, y) be any point on the curve. The

slope of AP is

$$\frac{y-0}{x+2} = \frac{\pm x\sqrt{x+2}}{x+2} = \frac{\pm x}{\sqrt{x+2}}.$$

As P approaches A, x approaches -2 and the slope of AP increases indefinitely. The curve is therefore perpendicular to the x-axis at A. The slope of OP is

$$\frac{y}{x} = \frac{\pm x\sqrt{x+2}}{x} = \pm\sqrt{x+2}.$$

As P approaches the origin, x approaches 0 and the slope of OP approaches $\pm\sqrt{2}$. At the origin the curve therefore makes with the x-axis the angles $\tan^{-1}(\pm\sqrt{2})$.

Exercises

Make graphs of the following equations:

1. $y = x(x+1)$.
2. $x = y^2(y-2)$.
3. $y = (x-1)(x+2)(x-3)$.
4. $y + 1 = x^4 + 2x$.
5. $y = x(x+1)(2x-3)$.
6. $y - 3 = x^2(x+1)(2x-3)$.
7. $y + 2 = x^3(x+1)^2(2x-3)$.
8. $y = x^2(x+1)^2(2x-3)^2$.
9. $y = x^3 - 1$.
10. $x - 2 = (y+1)^4 - 1$.
11. $x = y^4 + y^2 + 1$.
12. $x = y^4 + y^3 - 4y^2 - 4y$.
13. $y^2 = x(x+1)$.
14. $y^2 = (x^2-1)(x^2-4)$.
15. $(y+1)^2 = (x^2-1)(4-x^2)$.
16. $x^2 = (y-1)^2(y-2)$.
17. $x^2 = (y-1)^2(2-y)$.
18. $(y^2+x)(y^2-x) = 0$.
19. $y^2 + 2y = x^4 + 2x - 1$.
20. $(x+y)^4 - y^4 = 0$.
21. $(y+x)^2 = x^2(x-1)$.
22. $x^4 + y^4 = 1$.
23. $y^4 - 2xy^2 + x^2 - y^2 + 4 = 0$.
24. $y^4 + (2x^2+1)y^2 + x^4 - x^2 = 0$.
25. $x^3 + xy^2 = 1$.
26. $x^2 - 4xy + 8y^2 - y^4 = 0$.
27. $y = \dfrac{16}{1-x}$.
28. $y = \dfrac{x}{x+1}$.
29. $x = \dfrac{2y^2+3y-2}{y-3}$.
30. $y = \dfrac{(x+1)(x-2)}{x(x-3)}$.
31. $y = x + \dfrac{1}{x}$.
32. $y = x^2 + \dfrac{1}{x^2}$.
33. $x^2 = \dfrac{y^2-1}{y^2-4}$.
34. $y = \dfrac{1}{x-1} - \dfrac{1}{x+3}$.
35. $y = x^3 - \dfrac{1}{x(x-2)}$.
36. $y = \dfrac{1}{(x-1)^2} - \dfrac{1}{(x+3)^2}$.
37. $y^2 = \dfrac{1}{x(x-1)} - \dfrac{1}{x+3}$.

38. $x^2y^2 + 36 = 4\,y^2$.

39. $y^2 = \dfrac{x^2\,(a + x)}{a - x}$.

40. $x^3y + a^3y - x^4 + a^4 = 0$.

41. $y^3 = x^4$.

42. $x^4 = y^6$.

43. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

44. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

45. $x^2 + y^{\frac{2}{3}} = 1$.

46. $x^{13} + y^{23} = 1$.

47. $x^{12} + y^{22} = 1$.

48. $y^3 = x^2\,(x + 2)$.

49. $(y + 2)^3 = (x - 1)\,(x^2 - 4)$.

50. $(x + y)^2 = y^2\,(y + 1)$.

51. $(x + y - 1)\,(2\,x - 4\,y - 2) = 4$.

52. $y^2 = x\,(x - y)\,(x + y)$.

53. $(x + y - 2)^2 = (2\,x - y - 1)^3$.

54. $x\,(x^2 + y^2 - 1) = 1$.

Art. 41. Sine Curves

When a, b, c are constants, the graph of the equation

$$y = a \sin (bx + c)$$

is called a *sine curve*. As shown in Fig. 41b, the graph consists of a series of waves all having the same length and height. The sine of an angle is never greater than 1 and so y is never greater than a. The constant a therefore measures the height of the waves. If x is increased by $2\,\pi/b$, the angle $bx + c$ is increased by $2\,\pi$ and y is not changed. This is the smallest constant that added to x will leave y unchanged. Hence $2\,\pi/b$ is the distance from any point of a wave to the corresponding point of the next wave. It is called the *wave length.*

The equation $y = a \cos (bx + c)$ also represents a sine curve; for

$$a \cos (bx + c) = a \sin \left(\frac{\pi}{2} - bx - c\right) = a \sin (b'x + c'),$$

if $b' = -b$, $c' = \frac{1}{2}\,\pi - c$. Thus a cosine curve is a sine curve with different constants. Also

$$y = A \sin (mx) + B \cos (mx)$$

can be reduced to the form $y = a \sin (mx + b)$ and so represents a sine curve.

In plotting sine curves angles should be expressed in circular measure. A circle being drawn with the vertex of an angle as center,

the circular measure of the angle is defined as the ratio of the intercepted arc to the radius (Fig. 41*a*), that is,

$$\text{Circular measure of angle} = \frac{\text{intercepted arc}}{\text{radius of circle}}.$$

An angle of 180° has a circular measure equal to $\pi = 3.14159 \ldots$, and other angles have proportional measures. For instance, the circular measure of 360° is 2π; 90°, $\frac{1}{2}\pi$; 60°, $\frac{1}{3}\pi$; 45°, $\frac{1}{4}\pi$; 30°, $\frac{1}{6}\pi$. An angle whose circular measure is 1 intercepts an arc equal to the radius. This angle, called a *radian*, is

$$\frac{180°}{\pi} = 57°.3- .$$

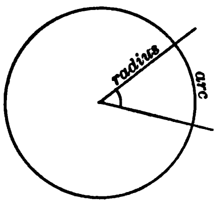

FIG. 41*a*.

If x is a real number, $\sin x$ means sine of the angle whose circular measure is x. Thus $\sin 2 = \sin (114°.6-)$.

Example 1. Plot the curve $y = 2 \sin (3x)$. The sine of an angle is never greater than 1 nor less than -1. Consequently, on this curve, y cannot be greater than 2 nor less than -2. The curve thus lies between the lines $y = -2$ and $y = 2$. The most important points on a sine curve are those where the sine is a maximum, minimum or zero. Between $3x = 0$ and $3x = 2\pi$, the important values are shown in the following table:

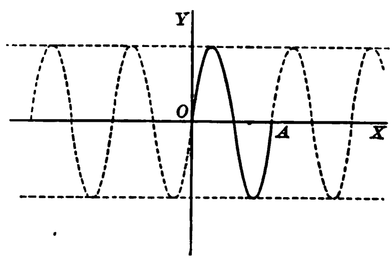

FIG. 41*b*.

$3x = 0$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π
$x = 0$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2}{3}\pi$
$y = 0$	2	0	-2	0

This part of the curve extends from O to A (Fig. 41*b*). When x is increased by $\frac{2}{3}\pi$, $3x$ is increased by 2π and y is not changed. Be-

yond A a new wave thus begins like that between O and A. In the same way it is seen that the wave on the left of O is like that from A to O. The whole curve thus consists of an infinite number of waves each of wave length $\frac{2}{3}\pi$.

Ex. 2. $x = \cos(2y - 3)$. Since the cosine is never greater than 1 nor less than -1, the curve lies between the lines $x = 1$, $x = -1$. At the point A (Fig. 41c) where $2y - 3 = 0$, x has the maximum value 1. Between this point and B, where $2y - 3 = 2\pi$, the most important values are shown in the following table:

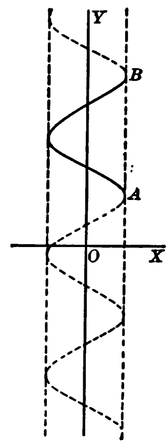

FIG. 41c.

$2y - 3 = 0$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π
$y = \frac{3}{2}$	$\frac{3}{2} + \frac{\pi}{4}$	$\frac{3}{2} + \frac{\pi}{2}$	$\frac{3}{2} + \frac{3}{4}\pi$	$\frac{3}{2} + \pi$
$x = 1$	0	-1	0	1

When y is increased or decreased by π, the change in the angle $2y - 3$ is 2π and x is not changed. The curve then consists of a series of waves like that from A to B placed end to end.

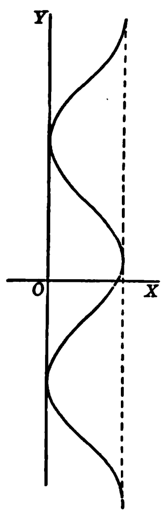

FIG. 41*d*.

Ex. 3. $y + 1 = \sin^{-1}(x - 1)$. This equation is equivalent to

$$x - 1 = \sin(y + 1).$$

The graph is a sine curve with axis $x - 1 = 0$. The curve passes through the point $y = -1$, $x = 1$, extending above and below that point in a series of waves each of vertical length 2π. (Fig. 41*d*.)

Ex. 4. $y = 2\sin(\frac{1}{2}x) + 6\cos(\frac{1}{4}x)$. To construct this curve, draw the curves

$$y_1 = 2\sin(\tfrac{1}{2}x), \quad y_2 = 6\cos(\tfrac{1}{4}x)$$

and on each vertical line determine the point P such that $y = y_1 + y_2$, that is,

$$MP = MP_1 + MP_2.$$

To leave $\sin(\frac{1}{2}x)$ and $\cos(\frac{1}{4}x)$ both unchanged, x must be increased by 8π or a multiple of 8π. Hence a section AB from $x = 0$ to

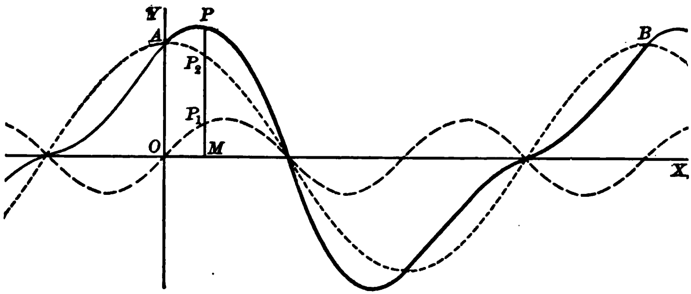

FIG. 41e.

$x = 8\pi$ must be plotted. The whole curve is a series of such sections placed end to end (Fig. 41e).

Art. 42. Periodic Functions

The equations considered in the previous article have the peculiarity that when a certain constant is added to one of the variables the other variable is not changed. Let the equation of a curve be $f(x, y) = 0$. If

$$f(x + k, y) = f(x, y), \quad \text{or} \quad f(x, y + k) = f(x, y),$$

the curve consists of a series of pieces (extending from x to $x + k$ or from y to $y + k$) each obtained by moving the preceding one a distance k in the direction of a coördinate axis. In this case the function $f(x, y)$ is called *periodic* and k is its *period.* The part of the curve from x to $x + k$ or from y to $y + k$ is called a *cycle.* For example, in Ex. 4 of the previous article a cycle extends from any point (x, y) of the curve to the point $(x + 8\pi, y)$. To plot a curve whose equation is periodic it is necessary to plot one cycle and sketch the others from periodicity.

Example 1. Plot the curve $y = \tan(2x)$. If the angle $2x$ is increased by π (x increased by $\frac{1}{2}\pi$) the tangent is not changed. The curve therefore consists of a series of parts each obtained by moving the preceding one a distance $\frac{1}{2}\pi$ to the right. One of these branches passes through the origin. As x increases from 0 to $2x = \frac{1}{2}\pi$, y increases from 0 to infinity. The line $x = \frac{1}{4}\pi$ is therefore an asymptote. As x decreases from 0 to $-\frac{1}{4}\pi$, y decreases to $-\infty$, the line $x = -\frac{1}{4}\pi$ being an asymptote. The branch through the origin thus extends from $x = -\frac{1}{4}\pi$, $y = -\infty$ to $x = \frac{1}{4}\pi$, $y = \infty$.

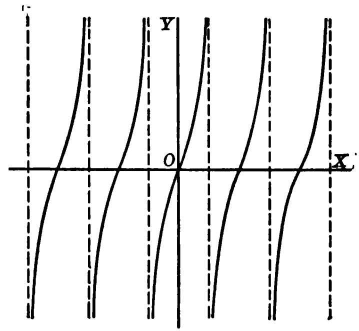

FIG. 42a.

The whole curve consists of a series of such branches at horizontal distances $\frac{1}{2}\pi$ apart (Fig. 42a).

Ex. 2. $y = \sec x$. The secant of an angle is in absolute value never less than 1. Consequently the curve lies outside the lines $y = \pm 1$. Since $\sec(-x) = \sec x$, the curve is symmetrical with respect to the y-axis. Since $\sec(x + 2\pi) = \sec x$, a complete cycle of the curve is contained between $x = 0$ and $x = 2\pi$. The most important points on this part of the curve are given in the following table:

$x = 0$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π
$y = 1$	$\infty, -\infty$	-1	$-\infty, \infty$	1

the expression $\infty, -\infty$ meaning that on the left of this point y is indefinitely large and positive but on the right indefinitely large and negative. The curve is a series of U-shaped branches alternately above $y = 1$ and below $y = -1$ (Fig. 42b).

Ex. 3. $x = \sec^2 y$. In this case x is never less than 1. The curve consists of a series of U-shaped branches on the right of $x = 1$ with asymptotes $y = \pm\frac{\pi}{2}$, $y = \pm\frac{3}{2}\pi$, etc. (Fig 42c).

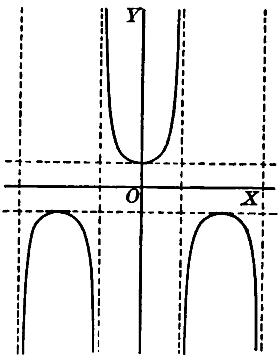

FIG. 42*b*.

Ex. 4. $y = \sin\left(\frac{1}{x}\right)$. The curve crosses the x-axis where

$$\frac{1}{x} = 0, \quad \pm\pi, \quad \pm 2\pi, \quad \pm 3\pi, \quad \ldots,$$

that is, where

$$x = \infty, \quad \pm\frac{1}{\pi}, \quad \pm\frac{1}{2\pi}, \quad \pm\frac{1}{3\pi}, \quad \ldots.$$

Between each pair of consecutive crossings it reaches one of the lines $y = \pm 1$. The curve has an infinite number of waves whose horizontal lengths approach zero near the y-axis (Fig. 42*d*).

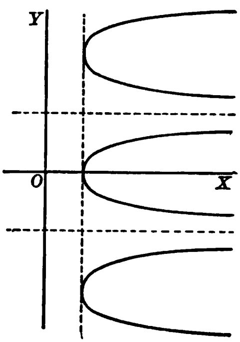

FIG. 42*c*.

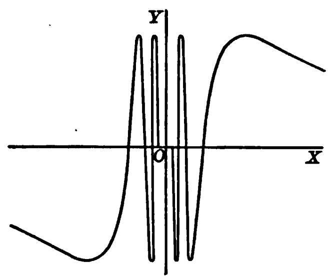

FIG. 42*d*.

Art. 43. Exponential and Logarithmic Curves

If a is a positive constant the function a^x is called an *exponential function.* It is understood that if x is a fraction a^x is the positive root.

If $y = a^x$ then, by definition, x is the logarithm of y to base a. Thus the equations

$$y = a^x, \quad x = \log_a y$$

are equivalent and both represent the same curve.

A particular number of great importance is

$$e = 1 + \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \cdots = 2.7182 \ldots .$$

Logarithms with e as base are called natural logarithms. The functions e^x and $\log_e x$ are much used in higher mathematics.

In working with exponentials and logarithms the following facts are often useful:

(1) $a^0 = 1$, $\log_a 1 = 0$, if a is not zero or infinite,

(2) $a^\infty = \infty$, $a^{-\infty} = 0$, $\log_a \infty = \infty$, $\log_a 0 = -\infty$, if $a > 1$,

(3) $a^\infty = 0$, $a^{-\infty} = \infty$, $\log_a \infty = -\infty$, $\log_a 0 = \infty$, if $a < 1$.

Example 1. Plot the curve $y = 2^x$, or $x = \log_2 y$. The curve crosses the y-axis at (0, 1). Since y is always positive the curve lies entirely above the x-axis. As x decreases to $-\infty$, y approaches zero and the curve approaches the x-axis which is an asymptote. As x increases, y increases. The increase in y for a given increase in x is greater the larger x; for, if x changes to $x + h$, the change in y is

$$2^{x+h} - 2^x = 2^x (2^h - 1),$$

and this is larger for larger values of x. If then x is increased by equal amounts h, the changes in y will form a series of steps of increasing height. The curve is thus concave upward and becomes more and more inclined to the x-axis as x increases (Fig. $43a$).

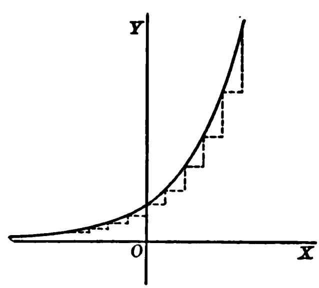

FIG. $43a$.

Ex. 2. $y = \log_{10}\left(\dfrac{x-1}{x+1}\right)$ or $x = \dfrac{1 + 10^y}{1 - 10^y}$. The logarithm of a negative number is imaginary. Hence y is real only when $x > 1$ or $x < -1$. When $x = 1$,

$$y = \log_{10} 0 = -\infty.$$

When $x > 1$, $(x - 1)/(x + 1)$ is less than 1 but approaches 1 as x increases indefinitely. Consequently y is negative but approaches 0 as x increases indefinitely. This part of the curve is thus below the x-axis and has the x-axis and the line $x = 1$ as asymptotes. When $x = -1$, $y = \log_{10}\infty = \infty$. If $x < -1$, $(x - 1)/(x + 1)$ is greater than 1 but approaches 1 as x decreases indefinitely. This part of the curve is therefore above the x-axis and has the x-axis and the line $x = -1$ as asymptotes (Fig. 43*b*).

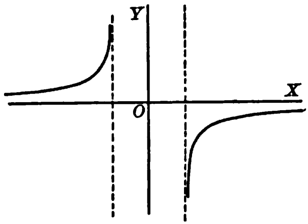

FIG. 43*b*.

Ex. 3. $y = \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$. When x is negative

$$e^{\frac{1}{x}} = \frac{1}{e^{-\frac{1}{x}}} < 1.$$

Consequently y is then negative and the curve is below the x-axis. When x is positive, y is positive and the curve is above the x-axis. As x approaches 0 from the negative side, $e^{\frac{1}{x}}$ approaches $e^{-\infty} = 0$ and y approaches -1. As x approaches 0 from the positive side, $e^{\frac{1}{x}}$ approaches infinity and y, being the ratio of two very large numbers whose difference is 2, approaches 1. Hence, as x passes through zero from the negative to the positive side, the point (x, y) jumps from $(0, -1)$ to $(0, +1)$. The curve is *discontinuous* at $x = 0$. When x becomes very large, whether it is positive or negative, y approaches zero. The x-axis is therefore an asymptote (Fig. 43*c*.)

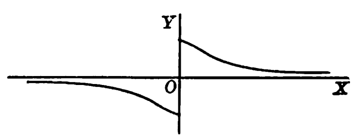

FIG. 43*c*.

Exercises

Plot the graphs of the following equations:

1. $y = \sin(2x)$.
2. $x = 2\sin(\frac{1}{2}y)$.
3. $y = 4\cos\left(x - \frac{\pi}{4}\right)$.
4. $y = 2\cos x + 3\sin x$.
5. $x = \sin y + \sin(2y)$.
6. $y = \cos(x-1) + \sin(x+1)$.
7. $y - 2 = \cos(2x+1)$.
8. $y = \sin^2 x$.
9. $y + 1 = \cos^{-1}(x-3)$.
10. $y = 2\tan(3x)$.
11. $x = \tan\left(y + \frac{\pi}{4}\right)$.
12. $y - 1 = \cot(x-3)$.
13. $x = \cot^{-1}(2y)$.
14. $y = \tan^2 x$.
15. $y^2 = \tan x$.
16. $y = \sec(2x)$.
17. $x = 1 + \csc y$.
18. $y = \sec x + \tan x$.
19. $y = \sec x \csc x$.
20. $y = x\sin x$.
21. $y = \cos\left(\frac{1}{x}\right)$.
22. $y = x\sin\left(\frac{1}{x}\right)$.
23. $y = e^x$.
24. $x = 2^{-y}$.
25. $y = 10^{x^2}$.
26. $y = e^{-x}\sin x$.
27. $x = \frac{1}{2}(3^y - 3^{-y})$.
28. $y = \frac{1}{2}(e^x + e^{-x})$.
29. $y\log_{10} x = 1$.
30. $y = \log_e[x(x-2)]$.
31. $y = \dfrac{10^{\frac{1}{x}}}{10^{\frac{1}{x}} - 1}$.
32. $y = \dfrac{1}{x}e^{-\frac{1}{x}}$.

Art. 44. Empirical Equations

Pairs of corresponding values of two variable quantities being given, it is sometimes desirable to find an equation connecting them. Let the pairs of values be plotted and draw a curve through the resulting points. However many points are given, the section of the curve between consecutive points can be arbitrarily drawn. Consequently an infinite number of curves can be drawn through the points. Each curve has an equation. An infinite number of equations are then satisfied by the given pairs of values. From a table of corresponding values it is not then possible to find the exact equation connecting the quantities.

It is usually assumed that if a smooth curve is drawn through or near the points, its equation will represent approximately the rela-

tion of the two quantities. Such an approximate equation is called *empirical*. A table of values being given, an infinite number of empirical equations are approximately satisfied by these values. The simplest equation should be chosen that has the required degree of accuracy. The choice of such an equation is largely a matter of judgment.

The following are types of equations it may be well to consider:

(1) $y = mx + b$.
(2) $y = ax^2 + bx + c$.
(3) $y = ax^n$.
(4) $y = ab^x$.

For convenience of plotting different unit lengths are often used along the two axes. This amounts to a uniform contraction in the direction of an axis. Equation (1) then still represents a straight line and (2) a parabola, but the constants in the equations have a different geometrical meaning.

Example 1. From the following values find an approximate equation connecting x and y:

$x = 0$	2.2	5.0	7.0	10	15
$y = 3.3$	4.0	6.0	6.5	8.4	11

These values are plotted in Fig. 44*a*. The points are seen to lie almost on a line. A line is drawn so that the points are about equally distributed on the two sides. This line crosses the y-axis at about (0, 3) and the line $x = 10$ at approximately (10, 8.4). The equation of the line through these points is

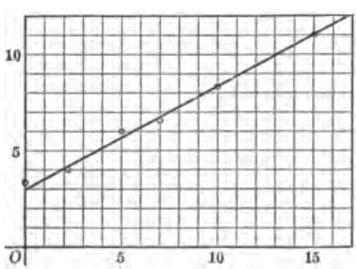

FIG. 44*a*.

$$y = 0.54\,x + 3,$$

which is the empirical equation required.

Ex. 2. Measurements of train resistance are given in the following table, where V = miles per hour, R = resistance in pounds per ton.

$V =$	20	40	60	80	100	120
$R =$	5.5	9.10	14.9	22.8	33.3	46.

The curve (Fig. 44*b*) looks like a parabola with horizontal axis. Let

$$R = A + BV + CV^2.$$

There being three coefficients, A, B, C, in this equation, the curve can be made to pass through only three of the given points. By more advanced methods the parabola could be found which fits closest to all the points. If a parabola is passed through three properly chosen points it will, however, usually be accurate enough. The points chosen should be spread over the whole curve and should not include any that appear to be faulty. In the present case numbers 1, 3 and 5 will be used. Substituting the coördinates of these points in the equation of the parabola,

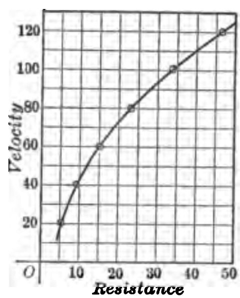

FIG. 44*b*.

$$\begin{aligned} 5.5 &= A + 20B + 400C, \\ 14.9 &= A + 60B + 3{,}600C, \\ 33.3 &= A + 100B + 10{,}000C. \end{aligned}$$

The solution of these equations is

$$A = 4.18, \quad B = 0.01, \quad C = 0.0028.$$

The empirical equation found is then

$$R = 4.18 + 0.01\,V + 0.0028\,V^2.$$

The curve (Fig. 44*b*) drawn from this equation is seen to pass very close to all the points.

Ex. 3. In the table below are given the loads which cause the failure of long wrought-iron columns with round ends, in which P/a is the load in pounds per square inch and l/r is the ratio of the

length of the column to the least radius of gyration of its cross-section.

l/r	P/a	$\log (l/r)$	$\log (P/a)$
140	12,800	2.1461	4.1072
180	7,500	2.2553	3.8751
220	5,000	2.3424	3.6990
260	3,800	2.4150	3.5798
300	2,800	2.4771	3.4472
340	2,100	2.5315	3.3222
380	1,700	2.5798	3.2304
420	1,300	2.6232	3.1139

Try the formula $P/a = C\,(l/r)^n$. Taking logarithms of both sides,

$$\log (P/a) = \log C + n \log (l/r).$$

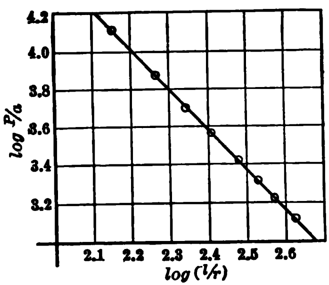

FIG. 44c.

This is a first degree equation connecting $\log (P/a)$ and $\log (l/r)$. If the formula is correct, the logarithms should then be coördinates of points on a line. Values of the logarithms are plotted in Fig. 44c. The points are almost on a line joining the first and last. The equation of this line is

$$\log (P/a) = -2.1 \log (l/r) + 8.62.$$

Consequently,

$$P/a = 417{,}000{,}000/(l/r)^{2.1},$$

which is the empirical equation required. This is approximately Euler's formula for the axial unit-load P/a which will cause a long wrought-iron column with round ends to fail.

Ex. 4. The following values were found for the amplitude of vibration of a long pendulum. Here A = amplitude in inches and t = time in minutes since the pendulum was set swinging.

$t =$	0	1	2	3	4	5	6
$A =$	10	4.97	2.47	1.22	0.61	0.30	0.14
$\log A =$	1	0.696	0.393	0.086	−0.215	−0.523	−0.854

Assume an equation of the form $A = ab^t$. Then

$$\log A = \log a + t \log b.$$

Using t and $\log A$ as coördinates the points should lie on a line. Fig. 44d shows this to be the case. The line seems to pass through the points $t = 0$ and $t = 5$. Its equation is then

$$\log A = 1 - 0.305\, t.$$

Consequently,

$$\log a = 1, \quad \log b = -\, 0.305,$$

and so $a = 10$, $b = 0.495$. The equation required is therefore

$$A = 10\,(0.495)^t.$$

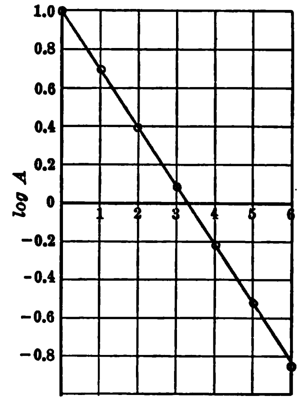

FIG. 44d.

Exercises

From the data in each of the following examples find an empirical equation connecting the quantities measured.

1. Test on square steel wire for winding guns. The stress is measured in pounds per square inch, the elongation in inches per inch.

Stress	Elongation	Stress	Elongation
5,000	0.00000	60,000	0.00216
10,000	0.00019	70,000	0.00256
20,000	0.00057	80,000	0.00297
30,000	0.00094	90,000	0.00343
40,000	0.00134	100,000	0.00390
50,000	0.00173	110,000	0.00444

2. Test on a steel column. The stress is measured in pounds per square inch, the compression in inches per inch.

Stress	Compression	Stress	Compression
3,000	0.00004	15,000	0.00039
6,000	0.00011	18,000	0.00053
9,000	0.00020	21,000	0.00066
12,000	0.00030	24,000	0.00087

3. The melting point θ, in degrees Centigrade, of a lead and zinc alloy containing x per cent lead, is given in the following table.

$x =$	40	50	60	70	80	90
$\theta =$	186	205	226	250	276	304

4. The following table gives the electromotive force E, in microvolts, produced in a lead and cadmium thermo-electric couple when the difference in temperature between the junctions is $\theta°$ C.

$\theta =$	-200	-100	0	100	200	300
$E =$	50	-140	0	475	1300	2425

5. The following table gives the number of grams S of anhydrous ammonium chloride which dissolved in 100 grams of water make a saturated solution at $\theta°$ absolute temperature.

$\theta =$	273	283	288	293	313	333	353	373
$S =$	29.4	33.3	35.2	37.2	45.8	55.2	65.6	77.3

6. The hysteresis losses in soft sheet iron subjected to an alternating magnetic flux are given in the following table, where B is flux density in kilolines per square inch, and P is watts lost per cubic inch for one cycle per second.

$B =$	20	40	60	80	100	120
$P =$	0.0022	0.0067	0.0128	0.0202	0.0289	0.0387

7. The observed temperatures θ of a vessel of cooling water at times t, in minutes, from the beginning of observation are given in the following table:

$t =$	0	1	2	3	5	7	10	15	20
$\theta =$	92°	85.3°	79.5°	74.5°	67°	60.5°	53.5°	45°	39.5°

8. Measurements showing the decay in activity of radium emanation are given in the following table:

Time in hours =	0	20.8	187.6	354.9	521.9	786.9
Relative activity =	100	85.7	24.0	6.9	1.5	0.19

CHAPTER 6

POLAR COÖRDINATES

Art. 45. Definitions

We shall now define another kind of coördinates called *polar*. Let O (Fig. 45*a*) be a fixed point and OX a fixed line. The point

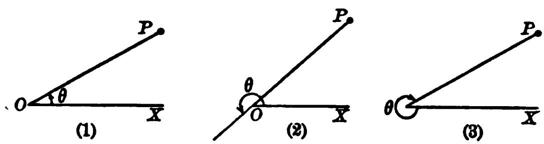

FIG. 45*a*.

O is called the *pole*, or origin, the line OX is called the *initial line*, or axis. The polar coördinates of a point P are the radius $r = OP$ and the angle θ from OX to OP.

The angle θ is any angle extending from OX to the line OP, the angle being considered positive when measured in the counter-clockwise direction (Fig. 45*a*, 1 or 2) and negative when measured in the clockwise direction (Fig. 45*a*, 3).

The radius r is considered positive when OP is the terminal side of θ (Fig. 45*a*, 1 or 3) and negative when θ terminates on OP produced.

A given point P is seen to have many pairs of polar coördinates, θ being any angle from OX to OP. A given pair of polar coördinates, however, determines a definite point obtained by constructing the angle θ and laying off r forward or backward along the terminal side according as r is positive or negative.

The point whose polar coördinates are r, θ is represented by the symbol (r, θ). To signify that P is the point (r, θ) the notation $P\ (r, \theta)$ is used.

Example 1. Plot the point $P\left(-1, -\frac{1}{4}\pi\right)$ and find its other pairs of polar coördinates.

The point is shown in Fig. 45*b*. The angle $-\frac{1}{4}\pi$ terminates on OP produced. The angle XOP is equal to $\frac{3}{4}\pi$. Any other angle terminating on OP or OP produced differs from one of these only by a positive or negative multiple of 2π. Any such angle then has one of the forms $2n\pi - \frac{1}{4}\pi$ or $2n\pi + \frac{3}{4}\pi$, where n is a positive or negative integer. Consequently any pair of polar coördinates of P has one of the forms

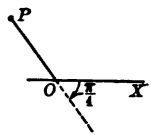

FIG. 45*b*.

$$\left(-1,\ 2n\pi - \frac{\pi}{4}\right), \qquad \left(1,\ 2n\pi + \frac{3}{4}\pi\right).$$

Ex. 2. Plot the point whose polar coördinates are (3, 4).

The coördinates of the point are

$$r = 3, \quad \theta = 4.$$

This means that the circular measure of θ is 4. The angle is approximately 229°. A line is drawn making this angle with OX and on the line the point P is located at a distance 3 from the origin (Fig. 45*c*).

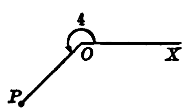

FIG. 45*c*.

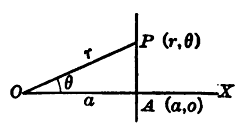

FIG. 45*d*.

Equation of a Locus. — The polar equation of a locus (like its rectangular equation) is satisfied by a pair of coördinates of every point on the locus and not satisfied by the coördinates of any point not on the locus.

Example 1. Find the polar equation of the line through A $(a, 0)$ perpendicular to the initial line.

Let P (r, θ) be any point on the line. From Fig. 45*d* it is seen that

$$r \cos \theta = a.$$

Conversely, any point whose coördinates satisfy this equation lies on the line, for the equation expresses that the projection of OP on OX is a.

Ex. 2. Find the polar equation of the circle whose diameter is the segment from the origin to the point A $(a, 0)$.

Let P (r, θ) be any point on the circle (Fig. 45*e*). In the right triangle OAP

$$r = OP = OA \cos\theta = a \cos\theta.$$

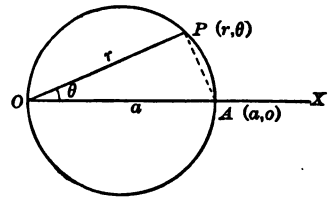

FIG. 45*e*.

Conversely, if $r = a \cos\theta$, the angle at P is a right angle and P lies on the circle. Hence $r = a\cos\theta$ is the equation required.

Exercises

1. Plot the points $(0, 30°)$, $(1, 40°)$, $\left(-2, \frac{\pi}{4}\right)$, $\left(3, -\frac{\pi}{3}\right)$, $(-4, -730°)$.

2. Plot the points whose polar coördinates are $(1, 1)$, $(-1, 2)$, $(2, -3)$, $(\sqrt{2}, \sqrt{3})$.

3. Show graphically that the points $(2\sqrt{2}, 0)$, $\left(2, \frac{\pi}{4}\right)$, $\left(2\sqrt{2}, \frac{\pi}{2}\right)$, $(\infty, \frac{3}{4}\pi)$ lie on a line. On this line what value of r corresponds to $\theta = \frac{5}{4}\pi$?

4. Show graphically that the points $(0, 0)$, $(\frac{3}{2}\sqrt{3}, 60°)$, $(3, 90°)$, $(-\frac{3}{2}, 210°)$ lie on a circle. What is its radius?

5. Show that $\left(1, \frac{\pi}{2}\right)$ and $\left(-1, -\frac{\pi}{2}\right)$ are the same point. Give other pairs of coördinates of this point.

6. Given the point P (r, θ), find the coördinates of the point Q if P and Q are symmetrical with respect to the initial line; symmetrical with respect to the origin; symmetrical with respect to the line through the origin perpendicular to the initial line.

7. Find the polar equation of the line through the origin making an angle of $\frac{\pi}{6}$ with the initial line. Does $\left(2, \frac{7}{6}\pi\right)$ lie on the line? Do its coördinates satisfy the equation?

8. Find the equation of the circle, radius a, with center at the origin.

9. Find the polar equation of the line parallel to the initial line and passing through the point $\left(a, \frac{\pi}{2}\right)$.

10. Find the polar equation of the circle, radius a, tangent to the initial line at the origin.

Art. 46. Change of Coördinates

The same point can be represented by rectangular or by polar coördinates. It is sometimes desirable to use both systems simultaneously. In this case the x-axis is usually coincident with the initial line and the origin of rectangular coördinates is the pole. A point P (Fig. 46a) then has four coördinates, x, y, r, θ. These are connected by several equations the most important of which are

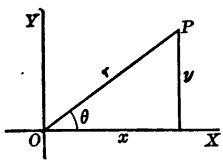

FIG. 46a.

$$\left.\begin{aligned} x &= r\cos\theta, & r &= \pm\sqrt{x^2+y^2}, \\ y &= r\sin\theta, & \tan\theta &= \frac{y}{x}. \end{aligned}\right\} \tag{46}$$

By the use of these equations (or, better, by the use of Fig. 46a) any expression in rectangular coördinates can be converted into one in polar coördinates and conversely.

Example 1. Find the polar equation of the circle $x^2 + y^2 = x + y$.

From Fig. 46a it is seen that

$$x^2 + y^2 = r^2, \quad x + y = r\cos\theta + r\sin\theta.$$

Hence the polar equation of the circle is

$$r^2 = r\cos\theta + r\sin\theta$$

or

$$r = \cos\theta + \sin\theta.$$

Ex. 2. By changing to rectangular coördinates show that

$$r\,(2\cos\theta + 3\sin\theta) = 4$$

is the polar equation of a straight line.

Since $r\cos\theta = x$, $r\sin\theta = y$, the rectangular equation is

$$2x + 3y = 4,$$

which is the equation of a line whose intercepts are $x = 2$, $y = \frac{4}{3}$.

Art. 47. Straight Line and Circle

Polar Equation of a Straight Line. — Let LK (Fig. 47a) be the line and let OD be the perpendicular upon it from the origin. Let

$$OD = p, \quad XOD = \alpha.$$

If $P\ (r, \theta)$ is any point on the line, $OP = r$, $XOP = \theta$. In the right triangle DOP, $OP \cos (DOP) = OD$, that is,

$$r \cos (\theta - \alpha) = p, \qquad (47a)$$

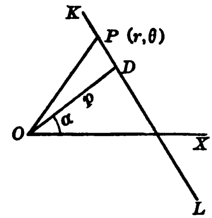

FIG. 47a.

which is the equation required.

Polar Equation of a Circle. — If the circle passes through the origin (Fig. 47b) let its radius be a and its center (a, α). Let A be the end of the diameter through the origin and let $P\ (r, \theta)$ be any point on the circle. In the triangle AOP,

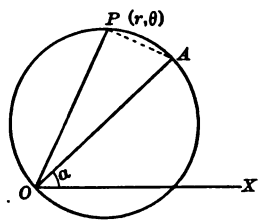

FIG. 47b.

$$OA = 2a, \quad OP = r, \quad AOP = \theta - \alpha.$$

Since $OP = OA \cos (AOP)$, the equation of the circle is then

$$r = 2a \cos (\theta - \alpha). \qquad (47b)$$

If the circle does not pass through the origin (Fig. 47c) let its radius be a and its center $C\ (b, \beta)$. Let $P\ (r, \theta)$ be any point on the circle. In the triangle COP, $CP^2 = OP^2 + OC^2 - 2\,OC \cdot OP \cos (COP)$, that is,

$$a^2 = r^2 + b^2 - 2\,br \cos (\theta - \beta),$$

which is the equation required.

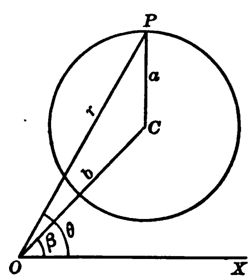

FIG. 47c.

Art. 48. The Conic

The locus of a point moving in such a way that its distance from a fixed point is proportional to its distance from a fixed straight line is called a conic. The fixed point is called a *focus*, the fixed line a *directrix* of the

conic. The constant ratio is called the *eccentricity* of the curve. The ellipse, parabola and hyperbola are all conics. The name conic refers to the fact that any section of a cone is a conic.

Let P be any point on a conic whose focus is F and directrix RS (Fig. 48*a*). If e is the eccentricity, the definition of the curve is

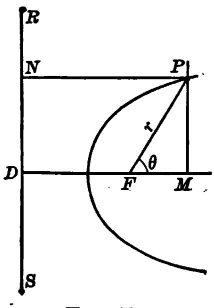

FIG. 48*a*.

$$FP = eNP.$$

Take F as origin and the line through F perpendicular to RS as the x-axis. Let $DF = k$. Then

$$FP = r, \quad NP = DF + FM = k + r\cos\theta.$$

Hence

$$r = e\,(k + r\cos\theta), \quad \text{or} \quad r = \frac{ke}{1 - e\cos\theta}. \tag{48a}$$

This is the equation of the conic with focus at the origin and directrix perpendicular to the initial line at the point $(-k, 0)$.

Change to rectangular coördinates, using F as origin and DF as x-axis. Then

$$r = \sqrt{x^2 + y^2}, \qquad r\cos\theta = x.$$

Equation (48*a*) can be written

$$r = ke + re\cos\theta = e\,(k + x).$$

Consequently,

$$x^2 + y^2 = e^2\,(k + x)^2. \tag{48b}$$

The Ellipse. — Suppose $e < 1$. Let a be greater than b and

$$e = \frac{\sqrt{a^2 - b^2}}{a}, \qquad k = \frac{b^2}{\sqrt{a^2 - b^2}}. \tag{48c}$$

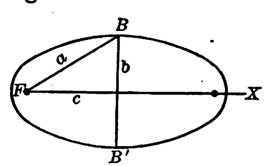

FIG. 48*b*.

Equation (48*b*) is then equivalent to

$$\frac{(x - \sqrt{a^2 - b^2})^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This is the equation of an ellipse with semi-axes a and b and major axis horizontal. Since a and b can have any values it follows, conversely, that any ellipse is a conic with eccentricity less than unity.

The center of the ellipse is $(\sqrt{a^2 - b^2}, 0)$. Since the origin is the focus the distance from the center to the focus is then

$$c = \sqrt{a^2 - b^2}. \tag{48d}$$

This shows that $FB = a$. By symmetry there is another focus at the same distance on the opposite side of the center. Hence an ellipse has two foci on the major axis at a distance from the ends of the minor axis equal to the semi-major axis.

The Parabola. — Suppose $e = 1$. Let $k = \frac{1}{2}a$. Equation (48b) is then equivalent to

$$y^2 = a\,(x + \tfrac{1}{4}a).$$

This is a parabola. Since a can have any value, it follows, conversely, that any parabola is a conic with eccentricity equal to unity. The vertex of the parabola is $(-\frac{1}{4}a, 0)$. The distance from the vertex to the focus is then equal to the absolute value of $\frac{1}{4}a$. Consequently, a parabola $y^2 = ax$ has a focus on its axis at the distance $\frac{1}{4}a$ from the vertex (Fig. 48c).

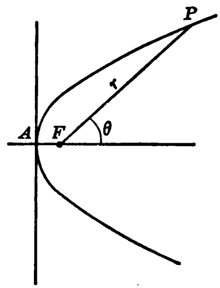

FIG. 48c.

The Hyperbola. — Suppose $e > 1$. Let

$$\left.\begin{aligned} e &= \frac{\sqrt{a^2 + b^2}}{a}, \\ k &= \frac{b^2}{\sqrt{a^2 + b^2}}. \end{aligned}\right\} \tag{48e}$$

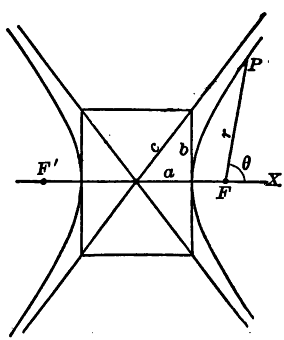

FIG. 48d.

Equation (48b) is then equivalent to

$$\frac{(x + \sqrt{a^2 + b^2})^2}{a^2} - \frac{y^2}{b^2} = 1.$$

This is the equation of a hyperbola with semi-axes a and b and transverse axis horizontal (Fig. 48d). Since a and b can have any values it follows, conversely, that any hyperbola is a conic with eccentricity greater than unity.

The center of the hyperbola is $(-\sqrt{a^2+b^2}, 0)$. Since the origin is the focus the distance from the center to the focus is then

$$c = \sqrt{a^2+b^2}. \tag{48f}$$

By symmetry there is another focus at the same distance on the other side of the center. Therefore, a hyperbola has two foci on its transverse axis at a distance from its center equal to half the diagonal of the rectangle on its axes.

Exercises

Determine the loci represented by the following equations. Construct the graphs. Change to rectangular coördinates.

1. $r \cos\theta = -2.$
2. $r \sin\theta = 3.$
3. $r = \sec\left(\theta + \frac{\pi}{4}\right).$
4. $\theta = -\frac{2}{3}\pi.$
5. $r = 3\cos\theta.$
6. $r = 4\sin\theta.$
7. $r = 2\sin\left(\theta - \frac{\pi}{4}\right).$
8. $r = \cos\theta - \sin\theta.$
9. $r = -2.$
10. $r^2 - 4r\cos\left(\theta - \frac{\pi}{3}\right) + 3 = 0.$
11. $r^2 - 2r(\cos\theta + \sin\theta) + 1 = 0.$
12. $r(2 - \cos\theta) = 2.$
13. $r(2 - 3\cos\theta) = 6.$
14. $r(1 - \cos\theta) = 3.$
15. $r(1 + \sin\theta) = -2.$
16. $r(1 + \sin\theta + \cos\theta) = 4.$
17. $r = 2\sec\theta - 3\csc\theta.$
18. $r\cos(2\theta) = \sin\theta.$

Determine the polar equations of the following loci:

19. $y = 2x - 1.$
20. $y^2 = 4x.$
21. $x^2 + y^2 = 2x.$
22. $xy = 7.$
23. $x^2 - y^2 = 1.$
24. $x^2 + y^2 = 4x + 4y.$

25. Find the polar equations of the circles of radius a tangent to both coördinate axes.

26. Find the polar equation of the parabola with focus at the origin and vertex $\left(-2, \frac{\pi}{6}\right)$.

27. Find the polar equation of the ellipse with focus at the origin, center on the y-axis, eccentricity $\frac{2}{3}$, and passing through the point $\left(1, \frac{\pi}{3}\right)$.

28. Find the eccentricity of the rectangular hyperbola $x^2 - y^2 = a^2$.

Art. 49. Graphing Equations

To plot the graph of a polar equation make a table of pairs of values satisfying the equation, plot the corresponding points and draw a smooth curve through them. It may be useful to look for any of the things mentioned in connection with plotting in rectangular coördinates. In most cases, however, it is sufficient to imagine θ, beginning with some definite value, to gradually increase and determine at each instant merely whether r is increasing or decreasing, draw a curve on which r varies in the proper direction and mark accurately the points where r is a maximum, minimum or

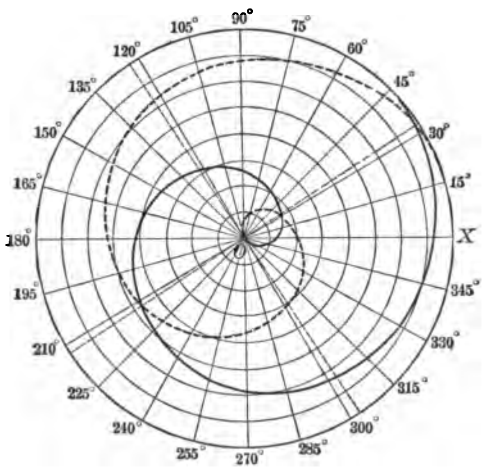

FIG. 49*a*.

zero. Proceed in the same way with θ decreasing from the initial value.

Example 1. $r = \theta + 1$. The curve passes through the origin at $\theta = -1$. As θ increases from this value, r is positive and steadily increases. While the angle makes a complete turn about the origin, r is increased by 2π. This part of the curve (indicated by the continuous line, Fig. 49a) thus consists of a series of expanding coils. Values of θ less than -1 give negative values of r (indi-

cated by the dotted line). Angles $\theta = -1 \pm k$ give numerically equal values of r. In particular, when k is an odd multiple of $\frac{1}{2}\pi$ the resulting points coincide. Hence the two sets of coils cross on the line perpendicular to $\theta = -1$. A curve like this containing an infinite number of coils is called a *spiral.*

Ex. 2. The cardioid $r = a(1 + \cos\theta)$.

The curve is shown in Fig. 49*b*. As θ increases from 0 to $\frac{1}{2}\pi$, r decreases from $2a$ to a. As θ increases from $\frac{1}{2}\pi$ to π, r decreases to 0. Since $\cos\theta$ cannot be less than -1, r does not become negative. As θ goes from π to 2π, r increases from 0 to $2a$. Since

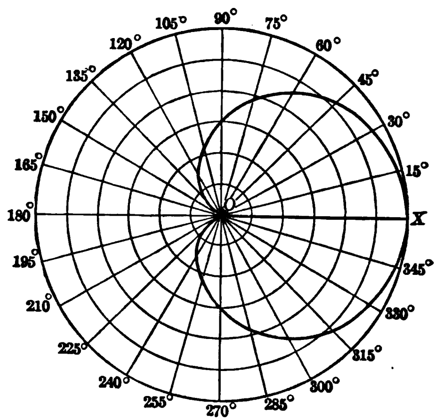

FIG. 49*b*.

$\cos(\theta + 2\pi) = \cos\theta$, values of θ greater than 2π and negative values give points already plotted.

Ex. 3. The lemniscate $r^2 = 2a^2\cos(2\theta)$. To each value of θ correspond two values of r differing only in sign. The curve is therefore symmetrical with respect to the origin. Also angles differing only in sign give the same values of r. Hence the curve is symmetrical with respect to the initial line. As θ increases from 0 to $\frac{1}{4}\pi$, r varies from $\pm a\sqrt{2}$ to 0. When θ is between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$, $\cos(2\theta)$ is negative and and r is imaginary. As θ increases from $\frac{3}{4}\pi$

to π, r goes from 0 to $\pm a\sqrt{2}$. Values of θ greater than π and negative values of θ give points already plotted (Fig. 49*c*).

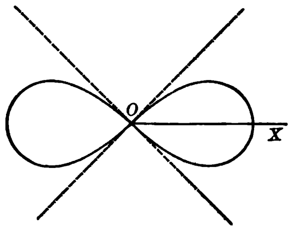

FIG. 49*c*.

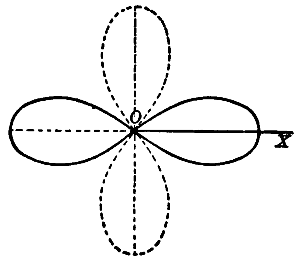

FIG. 49*d*.

Ex. 4. $r = a\cos(2\theta)$. The maximum, minimum and zero values of the cosine occur when the angle is zero or a multiple of $\frac{1}{2}\pi$. The most important points on the curve are then the following:

$$\begin{array}{cccccccccc}
2\theta = 0, & \frac{\pi}{2}, & \pi, & \frac{3}{2}\pi, & 2\pi, & \frac{5}{2}\pi, & 3\pi, & \frac{7}{2}\pi, & 4\pi, \\
\theta = 0, & \frac{\pi}{4}, & \frac{\pi}{2}, & \frac{3}{4}\pi, & \pi, & \frac{5}{4}\pi, & \frac{3}{2}\pi, & \frac{7}{4}\pi, & 2\pi, \\
r = a, & 0, & -a, & 0, & a, & 0, & -a, & 0, & a.
\end{array}$$

Values of θ greater than 2π give points already plotted (Fig. 49*d*).

Ex. 5. $r = a\sec(\frac{3}{2}\theta)$. If θ is increased by 4π the value of r is not changed. For

$$a\sec[\tfrac{3}{2}(\theta + 4\pi)] = a\sec(\tfrac{3}{2}\theta + 6\pi) = a\sec(\tfrac{3}{2}\theta).$$

The whole curve is then obtained by plotting points from $\theta = 0$ to $\theta = 4\pi$. The most important points are

$$\begin{array}{cccccccc}
\frac{3}{2}\theta = 0, & \frac{\pi}{2}, & & \pi, & \frac{3}{2}\pi, & & \dots, & 6\pi, \\
\theta = 0, & \frac{\pi}{3}, & & \frac{2}{3}\pi, & \pi, & & \dots, & 4\pi, \\
r = a, & +\infty, & -\infty, & -a, & -\infty, & +\infty, & \dots, & a,
\end{array}$$

where $+\infty$, $-\infty$ means that as θ increases to $\frac{1}{3}\pi$, r becomes indefinitely large and positive, while as θ decreases to $\frac{1}{3}\pi$, r becomes indefinitely large and negative. The curve is shown in Fig. 49*e*.

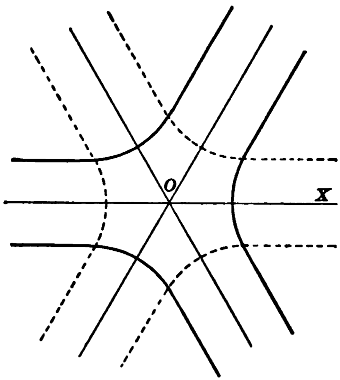

FIG. 49*e*.

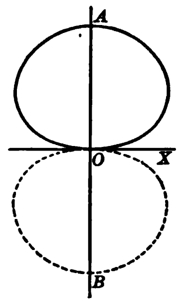

FIG. 49*f*.

Ex. 6. $r^2 = a^2 \sin(\theta)$.

The curve (Fig. 49*f*) is symmetrical with respect to the origin. When θ increases from 0 to $\frac{1}{2}\pi$, the positive value of r increases from 0 to a. As θ increases from $\frac{1}{2}\pi$ to π, r decreases to 0. Between $\theta = \pi$ and $\theta = 2\pi$ the sine is negative and no point is obtained. Negative values of θ and those greater than 2π give points already plotted.

Art. 50. Intersection of Curves

If the polar coördinates of a point satisfy the equation of a curve, the point lies on the curve. A point may, however, lie on a curve although its coördinates (as given) do not satisfy the equation of the curve. This happens because a point has several pairs of polar coördinates. One of these pairs may satisfy a given equation while another does not. Thus the point $B\left(1, -\frac{\pi}{2}\right)$ lies on the curve $r^2 = a^2 \sin(\theta)$ (Fig. 49*f*) but its coördinates do not satisfy the equa-

tion. The coördinates $\left(-1, \frac{\pi}{2}\right)$ represent the same point and satisfy the equation.

To find the intersections of two curves we solve their equations simultaneously. The pairs of coördinates thus obtained represent points on both curves. There may, however, be other points of intersection. This happens when some of the pairs of polar coördinates representing a point satisfy one equation, other pairs satisfy the other, but no pair satisfies both. In finding the intersections of curves represented by polar equations the graphs should always be drawn. Any extra intersections will then be seen.

Example 1. Show that the point $\left(a, \frac{\pi}{2}\right)$ lies on the curve $r^2 = a^2 \sin(3\,\theta)$.

The coördinates given do not satisfy the equation, for $\theta = \frac{\pi}{2}$ gives

$$r^2 = a^2 \sin\left(\tfrac{3}{2}\,\pi\right) = -a^2.$$

The point $\left(a, \frac{\pi}{2}\right)$ can, however, be written $\left(-a, \frac{3}{2}\pi\right)$ and in this form its coördinates satisfy the equation (Fig. 50*a*).

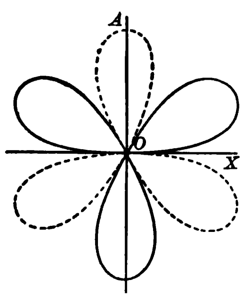

FIG. 50*a*.

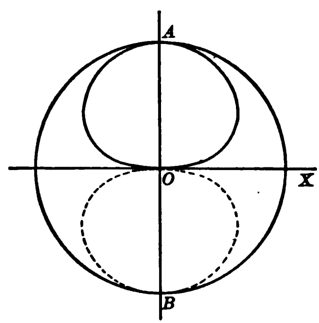

FIG. 50*b*.

Ex. 2. Find the intersections of the curves $r^2 = a^2 \sin\theta$, $r = a$. Solving simultaneously, we get

$$a^2 \sin\theta = a^2.$$

Consequently $\sin\theta = 1$ and $\theta = \frac{\pi}{2}$. One point of intersection is

then $A\left(a, \dfrac{\pi}{2}\right)$. It is seen from the figure that $B\left(-a, \dfrac{1}{2}\pi\right)$ is also a point of intersection.

Exercises

Plot the following pairs of curves:

1. $r = \sin\theta, \quad y = \sin x$.
2. $r = \cos\theta, \quad y = \cos x$.
3. $r = \tan\theta, \quad y = \tan x$.
4. $r = \sec\theta, \quad y = \sec x$.

Sketch the graphs of the following equations:

5. $r = \theta$.
6. $r = 1 - \theta$.
7. $r = \dfrac{1}{\theta}$.
8. $r = 2^{\theta}$.
9. $r = a\sin(2\theta)$.
10. $r = a\cos(3\theta)$.
11. $r = a(1 + \sin\theta)$.
12. $r = a(2 + \sin\theta)$.
13. $r = a(1 + 2\sin\theta)$.
14. $r = a\sin\left(\theta - \dfrac{\pi}{4}\right)$.
15. $r = a\cos(\tfrac{1}{2}\theta)$.
16. $r = a\tan(\tfrac{3}{2}\theta)$.
17. $r = a(1 + \sin 2\theta)$.
18. $r = a(1 + 2\cos 3\theta)$.
19. $r = 1 + \cos(\tfrac{3}{2}\theta)$.
20. $r = 4 + 5\cos(5\theta)$.
21. $r = 1 - 2\cos(\tfrac{4}{3}\theta)$.
22. $r = \sin\theta\cos\theta$.
23. $r = \sin^3\left(\dfrac{\theta}{3}\right)$.
24. $r^2 = 2a^2\sin(2\theta)$.
25. $r^2 = 1 + \sin\theta$.
26. $r^2 = \tan(2\theta)$.
27. $r = \dfrac{a}{\cos\theta} + \dfrac{a}{\sin\theta}$.
28. $r\sin\theta = a\cos(2\theta)$.
29. $r = \dfrac{2^{\frac{1}{\theta}} + 1}{2^{\frac{1}{\theta}} - 1}$.

30. Show that the point $(1, \tfrac{3}{4}\pi)$ lies on the curve $r = \sin(2\theta)$ but that the coördinates given do not satisfy the equation of the curve.

31. Show that the point $\left(1, \dfrac{3}{2}\pi\right)$ lies on the curve $r = -2\sin\left(\dfrac{\theta}{3}\right)$ but that the coördinates given do not satisfy the equation of the curve.

32. Show that if a is a constant the equations

$$r = \frac{1}{a\cos\theta + 1}, \qquad r = \frac{1}{a\cos\theta - 1}$$

represent the same curve.

33. Show that the equations $r^2 = a^2\cos^2(2\theta)$ and $r = a\cos(2\theta)$ represent the same curve.

34. Why do the equations $r^2 = ar\cos\theta$ and $r = a\cos\vartheta$ represent the same curve?

Plot the following pairs of curves and find their points of intersection:

35. $r = 2\sec\left(\theta - \dfrac{\pi}{4}\right), \qquad r = 2\sec\left(\theta + \dfrac{\pi}{3}\right)$.

36. $r \sin \theta = a, \quad r \cos \theta = a.$

37. $r^2 = a^2 \sin \theta, \quad r^2 = a^2 \sin (3\theta).$

38. $r = a \sin (2\theta), \quad r = a(1 - \cos 3\theta).$

39. $r^2 = 2a^2 \cos (2\theta), \quad r = a.$

Art. 51. Locus Problems

In finding the equation of the locus of a moving point, either polar or rectangular coördinates can be used. The system should be chosen which seems to fit the problem best, making a change of coördinates in the resulting equation if necessary. If the positions of origin and axes are not specified they should be placed in the most convenient position. This is usually (though not always) the most symmetrical position.

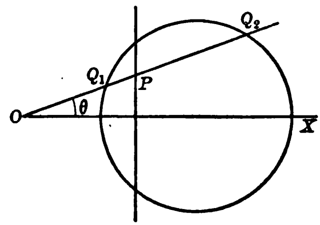

FIG. 51a.

Example 1. Through a fixed point O draw a line intersecting a fixed circle in Q_1 and Q_2 (Fig. 51a). On this line determine the point P such that

$$\frac{2}{OP} = \frac{1}{OQ_1} + \frac{1}{OQ_2}.$$

Find the locus of P as OP turns about O.

Take the origin at O and the initial line through the center $(b, 0)$ of the fixed circle. Let Q_1, Q_2, and P be (r_1, θ), (r_2, θ), and (r, θ) respectively. By hypothesis

$$\frac{2}{r} = \frac{1}{r_1} + \frac{1}{r_2}, \quad \text{or} \quad r = \frac{2r_1r_2}{r_1 + r_2}.$$

The equation of the fixed circle is $r^2 - 2rb \cos \theta + b^2 - a^2 = 0$, that is

$$r = b \cos \theta \pm \sqrt{a^2 - b^2 \sin^2 \theta}.$$

The two values of r in this equation are r_1 and r_2. Hence

$$r_1 + r_2 = 2b \cos \theta, \quad r_1r_2 = b^2 \cos^2 \theta - (a^2 - b^2 \sin^2 \theta) = b^2 - a^2.$$

Therefore

$$r = \frac{b^2 - a^2}{b \cos \theta} = \frac{b^2 - a^2}{b} \sec \theta$$

is the equation of the locus described by P. It is a straight line through the points at which lines through O are tangent to the circle.

Ex. 2. A point moves so that the product of its distances from two fixed points is equal to the square of half the distance between them. Find its locus.

Take the x-axis through the fixed points and the origin midway between them. Let the distance between the fixed points be $2a$. If $P(x, y)$ is any point on the locus (Fig. 51*b*), by hypothesis, $A'P \cdot AP = OA^2$, or

$$\sqrt{(x-a)^2+y^2}\,\sqrt{(x+a)^2+y^2} = a^2.$$

Squaring,

$$(x^2+y^2+a^2)^2 - 4a^2x^2 = a^4.$$

Changing to polar coördinates,

$$(r^2+a^2)^2 - 4a^2r^2\cos^2\theta = a^4,$$

or

$$\begin{aligned} r^2 &= 2a^2(2\cos^2\theta - 1) \\ &= 2a^2\cos(2\theta), \end{aligned}$$

which is the equation of a lemniscate.

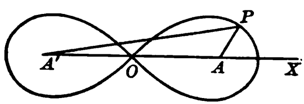

FIG. 51*b*.

Exercises

1. LK is a fixed straight line perpendicular to the initial line at $(a, 0)$. On any line through the origin, intersecting LK in Q, is taken a point P such that

$$OP \cdot OQ = a^2.$$

Find the locus of P as OQ turns about O.

2. LK is a fixed straight line perpendicular to the initial line at $A(a, 0)$. Any line through O intersects LK in Q and a point P is taken on this line such that

$$PQ = AQ.$$

Find the locus of P as OQ turns about O.

3. A point P moves in such a way that its distance from a fixed point O multiplied by its distance from a fixed straight line LK is constant. Find the locus of P.

4. A segment of fixed length slides with its ends in the x and y axes. Find the locus of the foot of the perpendicular from the origin to the moving segment.

5. A revolving line passing through the center of a fixed circle

intersects the circle in a point P_1 and a fixed straight line in P_2. Find the locus described by the point midway between P_1 and P_2.

6. Let OA be the diameter of a fixed circle and let LK be tangent to the circle at A. Through O draw any line intersecting the circle in D and LK in E. On OE lay off a distance OP equal to DE. Find the locus of P as OE turns about O.

7. From a point O on a fixed circle perpendiculars are dropped upon the tangents of the circle. Taking O as origin and the diameter through O as initial line find the polar equation of the curve generated by the feet of these perpendiculars.

8. A circle rolls along the initial line and a line through the center of the circle turns about the origin. Find the locus of the intersections of the moving line and circle.

9. A circle rolls along the initial line. A line through the origin moves in such a way as to remain tangent to the circle. Find the locus of the point of tangency.

10. Take a fixed point O and a fixed straight line BC. Through O draw any line intersecting BC in D and on this line lay off a constant distance DP measured from D in either direction. Find the locus described by P as the line turns about O.

11. Through a fixed point O on the circumference of a fixed circle draw any line cutting the circle again at D and lay off on this line a constant distance DP measured from D in either direction. Find the locus of P as OP turns about O.

12. A straight line OA of constant length revolves about O. Through A a line is drawn perpendicular to the initial line intersecting it in B. Through B a line is drawn perpendicular to OA intersecting it in P. Find the locus of P.

13. MN is a straight line perpendicular to the initial line at A (a, O). From O a line is drawn to any point B of MN. Through B a line is drawn perpendicular to OB intersecting the initial line at C. Through C a line is drawn perpendicular to BC intersecting MN at D. Finally, through D a line is drawn perpendicular to CD intersecting OB at P. Find the locus of P.

14. Two circles whose centers are fixed and whose circumferences touch rotate without slipping. A line through the center of one circle and rotating with it intersects a similar line on the other circle in a point P. Find the locus of P. If the radii are incommensurable show that the locus passes indefinitely near any point of the plane.

CHAPTER 7

PARAMETRIC REPRESENTATION

Art. 52. Definition of Parameter

In some cases it is more convenient to express the coördinates of a point on a curve in terms of a third variable than in terms of each other. Such a third variable is often called a *parameter* and the equations connecting the coördinates with the parameter are called *parametric equations*.

Example 1. A rod 5 feet long moves with its ends in the coördinate axes. Find the locus of the point P, 2 feet from the end in the x-axis.

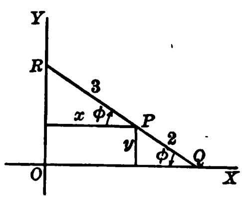

FIG. 52a.

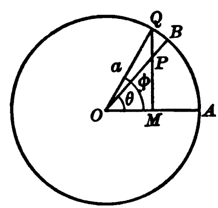

FIG. 52b.

Let QR be the rod and $\phi = \angle OQR$ (Fig. 52a). By hypothesis $PQ = 2$ and $PR = 3$. Let x, y be the coördinates of P. From Fig. 52a it is seen that

$$x = 3\cos\phi, \quad y = 2\sin\phi.$$

These are the equations expressing the coördinates of P in terms of the parameter. To find the equation connecting x and y we eliminate the parameter. The result is

$$\frac{x^2}{9} + \frac{y^2}{4} = \sin^2\phi + \cos^2\phi = 1.$$

The locus of P is therefore an ellipse.

Ex. 2. Find the locus of the point P, intersection of OB and MQ, (Fig. 52*b*) given that $MQ = \text{arc } AB$.

Let the radius of the circle be a and $\angle AOQ = \phi$. Let r, θ be the polar coördinates of P. Then

$$\text{arc } AB = a\theta, \quad MQ = a \sin \phi.$$

Consequently, $\theta = \sin \phi$. Also

$$r = \frac{OM}{\cos \theta} = \frac{a \cos \phi}{\cos (\sin \phi)}.$$

Polar parametric equations of the locus are therefore

$$\theta = \sin \phi, \quad r = a \cos \phi \sec (\sin \phi).$$

Elimination of ϕ gives the equation connecting r and θ in the form

$$r = a\sqrt{1 - \theta^2} \sec \theta.$$

Almost any quantity that varies from point to point of a curve can be used as a parameter. The equations connecting the coördinates with the parameter naturally depend on the parameter. There are then an infinite number of parametric equations of the same curve. The coördinate axes and the parameter must be fixed before the parametric equations have a definite form.

Exercises

1. A circle of radius a has its center at the origin. Express the coördinates x, y of any point P on the circle in terms of the angle ϕ between the x-axis and OP and so obtain parametric equations for the circle.

2. A $(0, a)$ is a fixed point on the y-axis and M a movable point on the x-axis. MP is perpendicular to AM and equal in length to it. Express the coördinates of P in terms of a and the angle ϕ from the right end of the x-axis to MP. By eliminating ϕ find the rectangular equation of the locus of P.

3. O is the center of a fixed circle tangent to AB at A. Through any point Q of AB passes another line tangent to the circle at R. On RQ determine the point P such that $RQ = QP$. Taking OA as x-axis and O as origin express the coördinates of P in terms of $\phi = \angle AOQ$. By eliminating the parameter ϕ determine the coördinate equation of the locus of P.

4. A segment AB through the point C $(2, 1)$ has its ends A and B in the x- and y-axes respectively. If P (x, y) is the middle point of AB

express x and y in terms of the angle $\phi = OAB$. By eliminating ϕ determine the rectangular equation of the curve described by P as AB turns about C.

5. A string, held taut, is unwound from a circle. Taking the origin at the center of the circle and the initial line through the point where the string begins to unwind, express the polar coördinates of the point P at the end of the string in terms of the radius of the circle and the angle at the center subtended by the arc unwound. Find the polar equation of the locus of P.

Art. 53. Locus of Parametric Equations

Suppose the coördinates x and y are given functions of a parameter ϕ,

(1) $$x = f_1(\phi), \qquad y = f_2(\phi).$$

If a value is assigned to ϕ the resulting values of x and y are coördinates of a certain point. The totality of such points will usually be found to form a curve. The above equations represent the curve in the sense that if any value be assigned to ϕ the resulting point lies on the curve and if any point (x, y) be taken on the curve there is a value of ϕ such that $x = f_1(\phi)$, $y = f_2(\phi)$.

To plot the curve we assign values to ϕ, considered as independent variable, calculate x and y and plot the resulting points.

To find the coördinate equation eliminate ϕ between the parametric equations. Let the result be

(2) $$f(x, y) = 0.$$

This equation, being a consequence of the parametric equations, is satisfied by the coördinates of any point on the curve. If, conversely, for every pair of values x, y satisfying (2) a value of ϕ can be found such that $x = f_1(\phi)$, $y = f_2(\phi)$, then (2) is the coördinate equation of the curve.

t	x	y
$\pm\infty$	$\pm\infty$	$\pm\infty$
± 2	± 2	± 8
$\pm\frac{3}{2}$	$\pm\frac{3}{2}$	$\pm\frac{27}{8}$
± 1	± 1	± 1
$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	$\pm\frac{1}{8}$
0	0	0

Example 1. Plot the curve whose parametric equations are $x = t$, $y = t^3$ and find its rectangular equation.

In the table values of x and y are placed after the corresponding values of the parameter t. The points represented by these pairs of coordinates are plotted (Fig. 53*a*) and a smooth curve drawn through

the resulting points. By eliminating t the rectangular equation of the curve is found to be $y = x^3$.

Ex. 2. $x = 3 \cos \phi$, $y = 2 \sin \phi$. While ϕ increases from 0 to $\frac{\pi}{2}$,

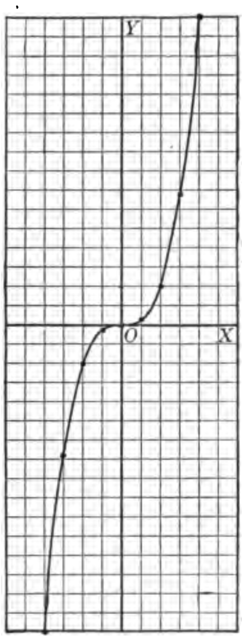

FIG. 53a.

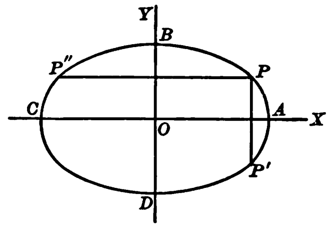

FIG. 53b.

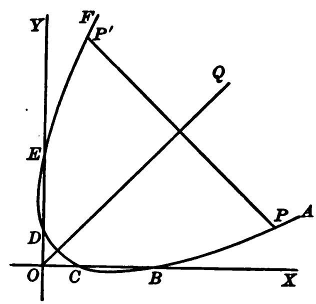

FIG. 53c.

x decreases from 3 to 0 and y increases from 0 to 2 (AB, Fig. 53b). As ϕ continues to π, y decreases to 0 while x decreases to -3 (BC, Fig. 53b). Then y decreases to -2 at $\phi = \frac{3}{2}\pi$ while x increases to 0. Finally, x increases to 3 and y to 0 at $\phi = 2\pi$. The symmetry with respect to the x-axis could have been foreseen since a change in the sign of ϕ makes a change in the sign of y but leaves x unchanged (P to P'). The symmetry with respect to the y-axis is indicated by the fact that a change from ϕ to $\pi - \phi$ changes the sign of x and makes no change in y (P to P'').

Ex. 3. $x = t(t-1)$, $y = (t+1)(t+2)$. The curve crosses the x-axis at $t = -1$ and $t = -2$ (C and B, Fig. 53c). It crosses

the y-axis at $t = 0$ and $t = 1$ (D and E). When t is a large negative number x and y are both positive. As t increases to -2, x and y both decrease (A to B). As t goes from -2 to -1, y is negative, x positive and decreasing (B to C). Between $t = -1$ and $t = 0$, y is positive and increasing, x positive and decreasing (C to D). Between $t = 0$ and $t = 1$, x is negative, y positive and increasing (D to E). When $t > 1$, x and y are both positive and increase with t (E to F). If t is replaced by $-t - 1$, the x and y coördinates interchange values (P to P'). Hence the curve is symmetrical with respect to the line OQ bisecting the angle between the axes. By subtraction

(1) $$y - x = 4t + 2, \quad \text{or} \quad t = \tfrac{1}{4}(y - x - 2).$$

Substituting t in the equation $x = t(t - 1)$ and simplifying

(2) $$(y - x)^2 - 8(y + x) + 12 = 0.$$

Conversely, if (1) and (2) are solved for x and y the parametric equations are obtained. Therefore (2) is the rectangular equation of the curve. It is a parabola.

Ex. 4. $x = \cos(2\phi)$, $y = \sin\phi$. When $\phi = 0$ we get the point $A(1, 0)$. As ϕ increases, x decreases and y increases until $\phi = \dfrac{\pi}{2}$ at $B(-1, 1)$. As ϕ continues to increase the point turns back, retraces the arc BA and continues to $C(-1, -1)$ where $\phi = \frac{3}{2}\pi$. As ϕ continues to increase the point oscillates back and forth along the path CAB (Fig. 53d).

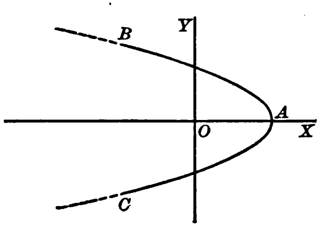

FIG. 53d.

Since $x = \cos(2\phi) = 1 - 2\sin^2\phi$ and $y = \sin\phi$,

$$x = 1 - 2y^2.$$

This equation is not however equivalent to the parametric equations. It represents a parabola extending to an infinite distance on the left, whereas the parametric equations (since $\sin\phi$ and $\cos 2\phi$ are never greater than 1) represent only the piece CAB.

Ex. 5. Find the intersections of the curves $x = a\cos\theta$, $y = b\cos(2\theta)$ and $x = a\sin\phi$, $y = b\sin(2\phi)$.

At a point of intersection both pairs of parametric equations are satisfied. Hence

$$\sin\phi = \cos\theta, \quad \sin(2\phi) = \cos(2\theta).$$

The general solution of the first of these equations is

$$\phi = \frac{\pi}{2} + 2n\pi - \theta$$

where n is any positive or negative integer. This value substituted in the second equation gives

$$\sin(2\theta) = \cos(2\theta).$$

Consequently $\tan(2\theta) = 1$ and $\theta = \dfrac{n\pi}{2} + \dfrac{\pi}{8}$. Hence

$$x = \pm a\cos\left(\frac{\pi}{8}\right), \quad \pm a\sin\left(\frac{\pi}{8}\right),$$

$$y = \quad b\cos\left(\frac{\pi}{4}\right), \quad -b\cos\left(\frac{\pi}{4}\right).$$

Art. 54. Parametric from Coördinate Equations

When the parametric equations of a curve are given its coördinate equation is obtained by eliminating the parameter. The converse problem is, given the coördinate equation and the definition of a parameter to find the parametric equations. To do this we use the definition of the parameter to obtain at least one equation connecting the coördinates and the parameter. This and the coordinate equation give two equations in three unknowns (two coördinates and a parameter). By solving these equations for the coördinates we obtain the parametric equations.

Example 1. Find parametric equations of the curve

$$x^3 + y^3 = xy,$$

using the ratio y/x as parameter.

Call the parameter t. Then by hypothesis

$$(1) \qquad \frac{y}{x} = t.$$

This and the equation of the curve, solved simultaneously, give

$$(2) \qquad x = \frac{t}{1+t^3}, \qquad y = \frac{t^2}{1+t^3}.$$

If we eliminate t from these equations we get the coördinate equation. Therefore the equations (2) are parametric equations of the curve; for if x, y are the coördinates of a point on the curve a value $t = y/x$ can be found such that x, y, t satisfy the parametric equations and conversely if x, y, t are any numbers satisfying the parametric equations then x, y are the coördinates of a point on the curve.

Ex. 2. Find the parametric equations of the curve

$$xy = 2x + 2y - 3,$$

using as parameter the slope of the line joining (x, y) and $(1, 1)$.

Call the parameter t. By hypothesis

$$(3) \qquad t = \frac{y-1}{x-1}.$$

This and the equation of the curve, solved simultaneously, give

$$(4) \qquad x = 2 + \frac{1}{t}, \qquad y = t + 2.$$

Conversely, elimination of t from (4) gives the equation of the curve. Consequently, the equations (4) are parametric equations of the given curve.

Ex. 3. Find parametric equations of the parabola

$$x = 1 - y^2,$$

using the parameter ϕ defined by the equation $y = \sin\phi$.

Substituting $y = \sin\phi$ in the equation of the curve we get $x = 1 - \sin^2\phi = \cos^2\phi$. The equations obtained are therefore

$$x = \cos^2\phi, \quad y = \sin\phi.$$

Since x and y given by these equations cannot be greater than 1, these are not parametric equations of the whole parabola but only of a piece of it. This is due to the fact that when y is numerically greater than 1 there is no parameter ϕ defined by the equation $y = \sin\phi$.

Exercises

Plot the curves represented by the following parametric equations and determine the corresponding coördinate equations:

1. $x = 1 + 2t, \quad y = 2 - 3t.$
2. $x = t^2, \quad y = 2t.$
3. $x = t + \dfrac{1}{t}, \quad y = t - \dfrac{1}{t}.$
4. $x = t^2(t-1), y = t^2(t+1).$
5. $x = \dfrac{t+1}{t-1}, \quad y = \dfrac{t}{t-2}.$
6. $x = \sin\phi, \; y = \sin\left(\phi + \dfrac{\pi}{6}\right).$
7. $x = \sec\phi, \quad y = \tan\phi.$
8. $x = t\sin t, \quad y = t\cos t.$
9. $r = \cos(2\phi), \quad \theta = \sin\phi.$
10. $r = t + \dfrac{1}{t}, \quad \theta = t - \dfrac{1}{t}.$
11. $r = \phi\sin\phi, \quad \theta = \phi\cos\phi.$
12. $x = a(\theta - \sin\theta), \; y = a(1 - \cos\theta).$
13. $r = \tan\phi, \quad \theta = \sin(3\phi).$
14. $x = a(3\cos\phi + \cos 3\phi),$ $y = a(3\sin\phi - \sin 3\phi).$

15. Plot two turns of the curve

$$x = \cos\phi, \quad y = \sin\left(\frac{7\pi\phi}{22}\right)$$

and show that the complete curve passes indefinitely near any point within a unit square.

16. Find the polar parametric equations of the curve

$$x = t\cos t, \quad y = t\sin t,$$

using the same parameter.

17. Find rectangular parametric equations of the curve

$$r = t^2, \quad \theta = 1 + t.$$

18. Show that $x = \sin t$, $y = \cos(2t)$ and $x = \sin(2t)$, $y = \cos(4t)$ are the same curve.

19. Are $x = t + \dfrac{1}{t}$, $y = t - \dfrac{1}{t}$ and $x = 2^t + 2^{-t}$, $y = 2^t - 2^{-t}$ the same curve?

20. Are $x = t + \dfrac{1}{t}$, $y = t - \dfrac{1}{t}$ and $x = \cos\theta + \sec\theta$, $y = \cos\theta - \sec\theta$ the same curve?

Find the intersections of the following pairs of curves:

21. $\left.\begin{array}{l} x = t^2 \\ y = 2t \end{array}\right\}, \; x^3 + y^3 = 8xy.$
22. $\left.\begin{array}{l} x = 5\cos\theta \\ y = 5\sin\theta \end{array}\right\}, \; \left.\begin{array}{l} x = 1 + t \\ y = 2 + t \end{array}\right\}.$
23. $\left.\begin{array}{l} x = a\cos\theta \\ y = a\sin\theta \end{array}\right\}, \; \left.\begin{array}{l} x = a\phi\cos\phi \\ y = a\phi\sin\phi \end{array}\right\}.$
24. $\left.\begin{array}{l} x = 2a\cos^2\phi \\ y = \dfrac{2a\cos^3\phi}{\sin\phi} \end{array}\right\}, \; r = 4a\cos\theta.$

25. Find rectangular parametric equations of the circle, radius a, with center on the x-axis and passing through the origin, using as parameter the polar coördinate θ.

26. Find parametric equations for the parabola

$$y^2 = 4x,$$

using the ratio $\frac{y}{x}$ as parameter.

27. Find parametric equations of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

using the parameter ϕ defined by the equation $x = a \cos \phi$.

28. Find parametric equations for the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

using the parameter ϕ defined by the equation $x = a \sec \phi$.

29. Find parametric equations of the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

using the parameter ϕ defined by the equation $x = a \sin^3 \phi$.

30. Find parametric equations of the curve

$$y^2 + 4y = x^3 - 2x^2 + x - 4,$$

using as parameter the slope of the line joining (x, y) and $(1, -2)$.

Art. 55. Locus Problems

The Cycloid. — When a circle rolls along a straight line a point of its circumference describes a curve called a cycloid.

Let a circle of radius a and center C roll along the x-axis (Fig. 55a). Let N be the point of contact with the x-axis and P (x, y)

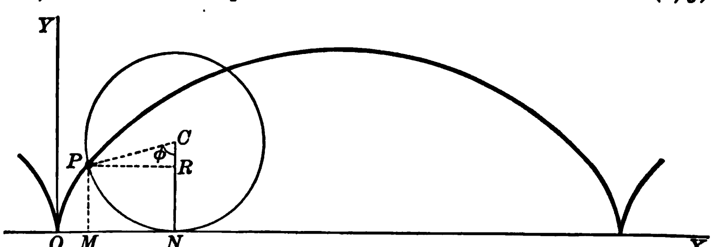

FIG. 55a.

the point tracing the cycloid. Take as origin the point O found by rolling the circle to the left until P meets OX. Take as parameter the angle $NCP = \phi$. Since the point of contact moves the same distance along the circle as along the straight line,

$$ON = \text{arc } NP = a\phi.$$

Consequently

$$x = OM = ON - MN = ON - PR = a\phi - a \sin \phi,$$
$$y = MP = NC - RC = a - a \cos \phi.$$

The parametric equations of the cycloid are then

$$x = a\,(\phi - a \sin \phi), \quad y = a\,(1 - \cos \phi).$$

The Epicycloid. — When a circle rolls on the outside of a fixed circle a point on its circumference describes an epicycloid.

Let a circle of radius a and center C roll on the outside of a circle of radius b and center O. Let N be the point of contact and $P\ (x, y)$ the point describing the epicycloid. Let A be the point obtained by rolling the moving circle backward until P meets the fixed circle.

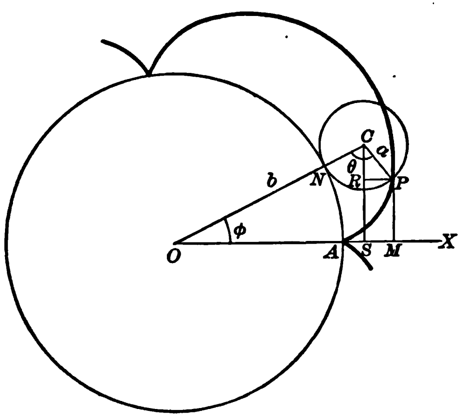

FIG. 55*b*.

Let O be the origin, OA the x-axis. Let $OCP = \theta$ and take as parameter the angle $AOC = \phi$. Since the point of contact moves the same distance along both circles arc AN = arc NP and consequently $b\phi = a\theta$. Also

$$RCP = OCP - OCR = \theta - \left(\frac{\pi}{2} - \phi\right) = \theta + \phi - \frac{\pi}{2}.$$

Therefore

$$x = OM = OS + RP = OC \cos \phi + CP \sin (RCP)$$
$$= (a + b) \cos \phi - a \cos (\theta + \phi) = (a + b) \cos \phi - a \cos \left(\frac{a+b}{a}\phi\right).$$

$$y = MP = SC - RC = OC \sin\phi - CP \cos(RCP)$$
$$= (a+b)\sin\phi - a\sin(\theta+\phi) = (a+b)\sin\phi - a\sin\left(\frac{a+b}{a}\phi\right).$$

The parametric equations of the epicycloid are then

$$x = (a+b)\cos\phi - a\cos\left(\frac{a+b}{a}\phi\right),\quad y = (a+b)\sin\phi - a\sin\left(\frac{a+b}{a}\phi\right).$$

Exercises

1. A circle of radius a moves with its center in the x-axis and a straight line passes through the center of the circle and a fixed point on the y-axis. Using as parameter the angle between the line and y-axis, find parametric equations for the curves traced by the intersections of moving line and circle.

2. Let AB be a given line, O a given point k units distant from AB. Draw any line through O meeting AB in M and let P be the point on this line such that

$$OM \cdot MP = k^2.$$

Find the parametric equations of the locus of P, using O as the origin, the perpendicular from O to AB as x-axis and the angle between OX and OP as parameter.

3. Let OA be the diameter of a fixed circle and LK the tangent at A. A variable line through O intersects the circle at B and LK at C. Through B draw a line parallel to LK and through C a line perpendicular to LK and call the intersection of these lines P. The locus of P is a curve called the witch. Find its parametric equations using the tangent at O as x-axis and the angle from the x-axis to OC as parameter. Also find the rectangular and polar equations of the curve.

4. Let O be the center of a circle, radius a, A a fixed point and B a moving point on the circle. If the tangent at B meets the tangent at A in C and P is the middle point of BC, find the equations of the locus of P using the angle AOB as parameter. Also find the rectangular equation.

5. $OBCD$ is a rectangle with $OB = a$, $BC = c$. Any line is drawn through C meeting OB in E and the triangle EPO is constructed so that the angles CEP and EPO are right angles. Find the locus of P, using the angle DOP as parameter, OB as x-axis and O as origin. Also find the rectangular equation of the locus.

6. A $(0, -a)$ and B $(0, a)$ are two fixed points on the y-axis. H is a variable point on the x-axis. BK is the perpendicular from B to AH meeting it in K. Through K a line is drawn parallel to the x-axis and through H a line is drawn parallel to the y-axis. These lines meet in P.

Find the equations of the locus of P using the angle BAK as parameter. Also find the rectangular and polar equations of the locus.

7. Let OA be the diameter of a fixed circle and LK the tangent at A. Through O draw any line intersecting the circle in B and LK in C and let P be the middle point of BC. Find the equations of the locus of P, using the angle AOP as parameter, OA as y-axis and O as origin. Find the rectangular and polar equations of the same curve.

8. OA is a diameter of a fixed circle and LK the tangent at A. Through O any line is drawn meeting the circle in B and LK in C. Through B a line is drawn perpendicular to OA meeting it in M. MB is prolonged to P so that $MP = OC$. Find the locus of P.

9. CD is perpendicular to OX and distant a units from O. A is a moving point on CD. AB is drawn perpendicular to OA meeting OX in B. BP is perpendicular to OX meeting OA in P. Find the locus of P.

10. Through a moving point B on a fixed circle lines from the ends of a diameter are extended a distance equal to the radius to form the sides of a square whose diagonal is BP. Find the locus of P.

11. The sides of a right angle are tangent to two fixed circles. Find the locus of the vertex.

12. Through two fixed points lines are drawn to form an isosceles triangle with its base in a fixed line. Find the locus of their point of intersection.

13. The angles of a triangle are A, B, C and the opposite sides are a, b, c. If the vertex A moves along the x-axis and B along the y-axis find the locus of C, using the angle between the side AB and the x-axis as parameter.

14. A string is wound around a circle and the end fastened at the center of the circle. A pencil resting against the string keeps it taut. Find the curve described as the string unwinds from the circle.

15. When a wheel rolls along a straight line any point in a spoke describes a trochoid. Let the wheel roll along the x-axis and use as parameter the angle ϕ in the equation of the cycloid. Find the parametric equations of the trochoid described by the point at distance b from the center of the circle.

16. A hypocycloid is the locus described by a point on the circumference of a circle which rolls internally on the circumference of a fixed circle. Find the parametric equations of the hypocycloid when the radius of the moving circle is $\frac{1}{4}$ that of the fixed circle, using a parameter analogous to that in the equations of the epicycloid.

17. A circle with center at the point $(2, 0)$ intersects a circle with center $(0, 2)$ in a point of the line $x = 3$. Find the locus of the other point of intersection of the two circles.

CHAPTER 8

TRANSFORMATION OF COÖRDINATES

It is sometimes desirable to move the axes to a new position. The coördinates will then be changed and it is necessary to find the new coördinates. There are two simple cases a combination of which gives any motion. These are translation, in which the origin is moved to a new position without changing the direction of the axes, and rotation, in which the origin is left fixed and the axes turned like a rigid frame about it.

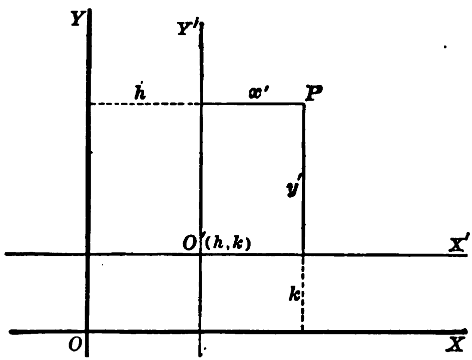

FIG. 56a.

Art. 56. Translation of the Axes

Let OX and OY (Fig. 56a) be the axes in their first position, $O'X'$ and $O'Y'$ the axes after motion. Let h, k be the coördinates of the new origin with respect to the old axes, x, y and x', y' the old and new coördinates of any point P. From Fig. 56a it is seen at once that

$$x = x' + h, \quad y = y' + k. \tag{56}$$

These are the equations connecting the new and old coördinates of any point.

Example 1. The origin is moved to the point $(-1, 2)$ the new axes being parallel to the old. Find the new equation of the curve

$$x^2 + 2x - y + 3 = 0.$$

The equations connecting new and old coördinates are in this case

$$x = x' - 1, \quad y = y' + 2.$$

When these values are substituted for x and y the equation of the curve becomes

$$x'^2 - y' - 0.$$

Ex. 2. Find the equation of the curve $r = a\cos\theta$ referred to a new pole at the point $\left(\frac{a}{\sqrt{2}}, \frac{\pi}{4}\right)$ the new initial line being parallel to the old.

The equations for transformation being given in rectangular coördinates, we change to rectangular coördinates, move the origin and then change back to polar coördinates. The coördinates of the new origin are

$$h = \frac{a}{\sqrt{2}}\cos\left(\frac{\pi}{4}\right) = \frac{a}{2}, \qquad k = \frac{a}{\sqrt{2}}\sin\left(\frac{\pi}{4}\right) = \frac{a}{2}.$$

The rectangular equations for transformation are therefore

$$x = x' + \frac{a}{2}, \qquad y = y' + \frac{a}{2}.$$

The rectangular equation of $r = a\cos\theta$ is

$$x^2 + y^2 = ax,$$

which transforms into

$$x'^2 + y'^2 + ay' = 0.$$

In polar coördinates this is

$$r' + a\sin\theta' = 0,$$

which is the equation required.

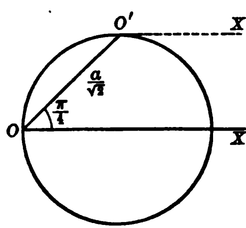

FIG. 56*b*.

Ex. 3. Transform the equation $x^3 + y^3 = 3x - 3y$ to new axes, parallel to the old, so chosen that there are no terms of first degree in the new equation.

Let $x = x' + h$, $y = y' + k$. The equation becomes

$$x'^3 + y'^3 + 3\,hx'^2 + 3\,ky'^2 + 3\,(h-1)\,x' + 3\,(k+1)\,y' + h^3 + k^3 - 3\,h + 3\,k = 0.$$

If there are no terms of first degree

$$h - 1 = 0, \quad k + 1 = 0.$$

Consequently, $h = 1$, $k = -1$ and the transformed equation is

$$x'^3 + y'^3 + 3\,x'^2 - 3\,y'^2 - 6 = 0.$$

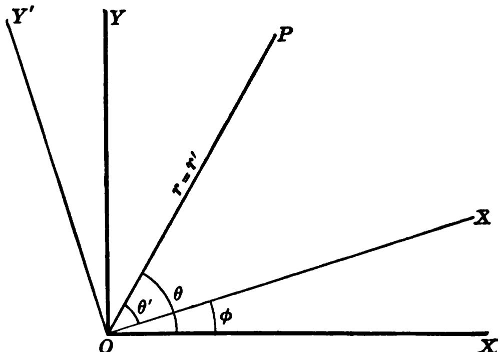

FIG. 57a.

Art. 57. Rotation of the Axes

The rotation is most easily expressed in polar coördinates. Let the initial line OX be rotated through an angle ϕ to OX' (Fig. 57a) the origin remaining fixed. Let the old and new coördinates of any point P be r, θ and r', θ'. From the figure it is seen that

$$r = r', \quad \theta = \theta' + \phi.$$

Using these relations we get

$$x = r\cos\theta = r'\cos(\theta' + \phi) = r'\cos\theta'\cos\phi - r'\sin\theta'\sin\phi,$$

$$y = r\sin\theta = r'\sin(\theta' + \phi) = r'\sin\theta'\cos\phi + r'\cos\theta'\sin\phi.$$

Replacing $r'\cos\theta'$ and $r'\sin\theta'$ by x' and y',

$$\left.\begin{aligned} x &= x'\cos\phi - y'\sin\phi, \\ y &= y'\cos\phi + x'\sin\phi. \end{aligned}\right\} \qquad (57)$$

Example 1. Find the new equation of the curve

$$x^3 + y^3 = x - y$$

after the axes have been rotated through an angle of 45°.

In this case $x = x'\cos(45°) - y'\sin(45°) = \dfrac{x' - y'}{\sqrt{2}}$,

$$y = y'\cos(45°) + x'\sin(45°) = \frac{x' + y'}{\sqrt{2}}.$$

Substituting these values and simplifying

$$x'^3 + 3\,x'y'^2 + 2\,y' = 0.$$

Ex. 2. Find the polar equation of the curve

$$x^2 - y^2 - 2xy\sqrt{3} = 2$$

after the origin is moved to the point $\left(-\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right)$ and the axes rotated through an angle of $-30°$.

The equations for moving the origin to $\left(-\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right)$ are

$$x = x' - \frac{\sqrt{6}}{2}, \qquad y = y' + \frac{\sqrt{2}}{2}.$$

If the axes are then rotated through $-30°$

$$x' = \frac{x''\sqrt{3} + y''}{2}, \qquad y' = \frac{y''\sqrt{3} - x''}{2}.$$

Consequently,

$$x = \frac{x''\sqrt{3} + y'' - \sqrt{6}}{2}, \qquad y = \frac{y''\sqrt{3} - x'' + \sqrt{2}}{2}.$$

These values substituted in the original equation give

$$x''^2 - 2\sqrt{2}\,x'' - y''^2 + 1 = 0.$$

Changing to polar coördinates

$$r^2(2\cos^2\theta - 1) - 2\sqrt{2}\,r\cos\theta + 1 = 0,$$

or

$$r = \frac{1}{\sqrt{2}\cos\theta \pm 1}.$$

The curves given by the positive and negative signs in this equation are identical. The equation required is then

$$r = \frac{1}{\sqrt{2}\cos\theta + 1}.$$

Art. 58. Invariants

Quantities associated with a curve are of two kinds, those depending on the position of the axes and those independent of the position of the axes. Quantities of the second kind are called *invariant.* For example, the radius of a circle is invariant (does not depend on the position of the axes) but the coördinates of its center are not invariant (change when the axes change position).

The equation of a curve is not invariant but there are some things connected with it that are invariant. One of the simplest is the degree of a rectangular equation. Wherever the axes are placed this degree is always the same. To see this it is only necessary to note that the equations for change of axes always have the form

$$x = ax' + by' + c, \qquad y = dx' + ey' + f,$$

where a, b, c, d, e, f are constants. A term $x^m y^n$ is equal to a sum of terms none of which are of higher degree than the $(m + n)$th in x' and y'. Hence through change of axes the degree of a polynomial cannot be increased. Neither can it be diminished for, in that case, changing back to the old axes would have to increase it. Therefore the degree of a rectangular equation is invariant.

If a rectangular equation can be factored so can the new equation resulting through a change of axes; for each factor of the old will be changed into a factor of the new. Also since real expressions are replaced by real, if either has real factors the other will have real factors also.

Art. 59. General Equation of the Second Degree

An equation of the second degree in rectangular coördinates has the form

(1) $$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Rotating the axes through an angle ϕ the coefficient of $x'y'$ in the resulting equation is

$$-2A\cos\phi\sin\phi + B(\cos^2\phi - \sin^2\phi) + 2C\cos\phi\sin\phi$$
$$= B\cos(2\phi) + (C - A)\sin(2\phi).$$

This is zero if

(2) $$\tan(2\phi) = \frac{B}{A - C}.$$

If then the axes are turned through an angle ϕ satisfying equation (2), equation (1) takes the form

(3) $$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0.$$

If A' and C' are both different from zero, completion of the squares gives an equation of the form

$$A'(x-h)^2 + C'(y-k)^2 = R.$$

If R is zero this equation is reducible. If R is not zero the equation can be written

$$\frac{(x'-h)^2}{R/A'} + \frac{(y'-k)^2}{R/C'} = 1.$$

The locus of this equation is an ellipse if the denominators are both positive, a hyperbola if one is positive and one negative, and is imaginary if both are negative.

Let $A' = 0$. Completion of the square in y' terms then gives

$$C'(y'-k)^2 = -D'(x'-h).$$

The locus of this equation is a parabola. Similarly if C' is zero the locus is a parabola. A' and C' cannot both be zero as the equation would then be one of the first degree.

We have thus found that *the second degree equation in rectangular coördinates is either reducible or else its locus is an ellipse, a parabola, a hyperbola, or an imaginary curve.*

Exercises

1. What are the rectangular coördinates of the points (2, 3), (−4, 5) and (5, −7) referred to parallel axes through the point (3, −2)?

2. Find the polar coördinates of the points $\left(3, \frac{\pi}{3}\right)$, $\left(-2, \frac{\pi}{2}\right)$, $\left(4, \frac{2\pi}{3}\right)$ if the origin is moved to the point $\left(2, \frac{\pi}{3}\right)$, the new initial line having the same direction as the old.

3. Find the equation of the curve

$$x^2 + 4y^2 - 2x + 8y + 1 = 0$$

referred to parallel axes through the point (1, −1).

4. Find the equation of the curve

$$y^3 - 6y^2 + 3x^2 + 12y - 18x + 35 = 0$$

referred to parallel axes through (3, 2).

5. Find the equation of the circle

$$r^2 - 5\sqrt{3}\,r\cos\theta - 5r\sin\theta + 21 = 0$$

when the origin is moved to the point $\left(5, \frac{\pi}{6}\right)$, the new initial line being parallel to the old.

6. Find the equation of the parabola

$$r = \frac{2p}{1 - \cos\theta}$$

when the pole is moved to $(-p, 0)$, the direction of the initial line being unchanged.

7. Find the equation of the line

$$2x - 3y + 7 = 0$$

referred to parallel axes through the point $(-5, -1)$.

8. Find the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

when referred to parallel axes through the left-hand vertex.

9. Find the equation of the witch

$$y = \frac{8a^3}{x^2 + 4a^2}$$

referred to parallel axes through the intersection of the y-axis and the curve.

10. Find the equation of the strophoid

$$y^2 = \frac{x^2(a - x)}{a + x}$$

referred to parallel axes through the intersection of the x-axis and the asymptote of the curve.

11. Find the parametric equations of the cycloid

$$x = a(\phi - \sin\phi), \quad y = a(1 - \cos\phi)$$

referred to parallel axes through the highest point of an arch of the curve.

12. Find parametric equations for the cardioid

$$r = a(1 + \cos\theta)$$

referred to parallel axes through the point $x = \frac{1}{2}a$, $y = 0$.

13. Transform the equations

$$x + y - 3 = 0, \quad 2x - 3y + 4 = 0$$

to parallel axes so chosen that the new equations have no constant terms.

14. Transform the equation

$$x^2 + y^2 - 2x + 4y - 6 = 0$$

to parallel axes so chosen that the new equation has no terms of first degree in x and y.

15. Show that by a change to parallel axes the equation

$$y^2 + 4y - 2x + 3 = 0$$

can be reduced to the form

$$y'^2 = 2x'.$$

16. Show that by moving the origin to the point $(p, 0)$ and changing to polar coördinates the equation

$$y^2 = 4px$$

can be reduced to the form

$$r' = \frac{2p}{1 - \cos\theta'}.$$

17. Show that by a change to parallel axes the equation

$$x^2 + 4y^2 - 3x + 5y = 6$$

can be reduced to the form

$$x'^2 + 4y'^2 = b^2,$$

where b is a constant.

18. By a change to parallel axes reduce

$$2x^2 - 6y^2 + 3x - 4y = 12$$

to the form $(x/a)^2 - (y/b)^2 = 1$.

19. What are the new coördinates of the points $(0, 1)$, $(1, 0)$, $(1, 1)$, $(-1, -1)$ after the axes have been rotated through an angle of 60°?

20. Transform the equation

$$x^2 + xy + y^2 = 1$$

to new axes bisecting the angles between the original axes.

21. Find the equation of the curve

$$r^2 \sin\theta \cos\theta = 1$$

referred to the same pole but with the initial line rotated through 45°.

22. By rotating the axes through the proper angle transform the equation $xy = 4$ to a form without xy-term.

23. By a proper change of axes reduce $x^2 + xy = 2$ to the form $(x'/a)^2 - (y'/b)^2 = 1$.

24. By a proper change of axes reduce the equation

$$x + 3y - 4 = 0$$

to the form $x' = 0$.

25. If the curve $y^2 = 4x$ is referred to new axes show that the second degree terms in the new equation form a complete square.

26. If the curve $x^2 + 4y^2 - 5 = 0$ is referred to new axes show that the second degree part of the new equation has imaginary factors.

27. If the curve $x^2 - y^2 - 1 = 0$ is referred to new axes show that the second degree part of the new equation has real factors.

CHAPTER 9

COÖRDINATES OF A POINT IN SPACE

Art. 60. Rectangular Coördinates

Let $X'X$, $Y'Y$ and $Z'Z$ be three scales with a common zero point O called the *origin*. The lines $X'X$, $Y'Y$ and $Z'Z$ are called *coördinate axes* and are referred to as the x-axis, y-axis and z-axis respectively. They determine three *coördinate planes* XOY, YOZ and ZOX called the xy-, yz- and zx-planes. These planes divide space into eight portions called *octants*. The portion O–XYZ is sometimes called the first octant. The other octants are not usually numbered.

Through any point P pass planes perpendicular to the coördinate axes meeting them in M, R and T. The numbers at these points in the scales are called the coördinates x, y, z of P. This point is represented by the symbol (x, y, z). To indicate the coördinates of P the notation $P\ (x, y, z)$ is used.

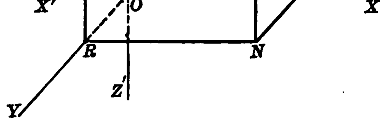

FIG. 60.

Usually the x-axis is drawn to the right, the y-axis forward and the z-axis upward. The x-coördinate of P is then the segment SP considered positive when drawn to the right, the y-coördinate is QP considered positive when drawn forward and the z-coördinate is NP considered positive when drawn upward.

To plot the point P having coördinates x, y, z draw $OMNP$ making $OM = x$, $MN = y$ and $NP = z$. The result is a plane figure.

By shading and dotting lines it can however be given the appearance of a space construction.

The points in a coördinate plane have one coördinate equal to zero. With respect to the other two coördinates these points can be treated as in plane geometry. For example, the points in the yz-plane whose coördinates y and z satisfy a first degree equation lie on a line.

Many results of solid geometry are similar to those already found in the plane. A formula in two coördinates x and y is often extended to space by adding a similar term containing z. With a little attention to these relations many formulas of space geometry can be inferred from analogy with those in the plane.

Art. 61. Projection

The projection of a segment AB upon a line RS is the segment of that line intercepted between planes through A and B perpen-

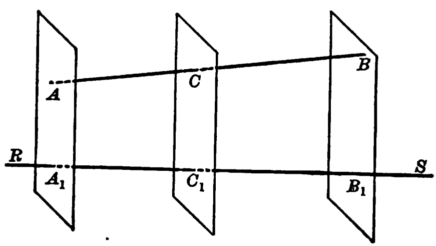

FIG. 61*a*.

dicular to RS. The projection of AB upon a plane is the segment between the feet of perpendiculars from A and B to the plane. Since parallel planes or lines intercept two fixed lines proportionally, it follows, as in plane geometry, that segments of a line have the same ratios as their projections on a line or plane. Thus, in Figs. 61*a* or 61*b*,

$$AB : BC = A_1B_1 : B_1C_1.$$

In using this proportion AB and BC must have algebraic signs if their projections have algebraic signs and conversely.

Projection on a Directed Line. — By the angle between two lines that do not intersect is meant the angle between intersecting lines parallel to them. Two lines determine two angles less than 180°, one acute and one obtuse. If the lines are directed they however determine a definite angle, the one less than 180° with the arrows on its sides pointing away from its vertex.

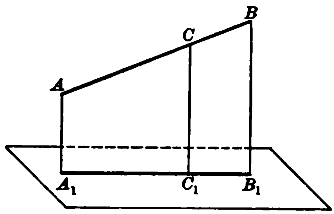

FIG. 61*b*.

Let θ be the angle between a directed segment AB and a directed line RS (Fig. 61*c*). Through A draw AB' parallel to RS to meet the plane through B perpendicular to RS. If the projection A_1B_1 is considered positive when it is drawn in the direction RS

$$A_1B_1 = AB' = AB\cos\theta,$$

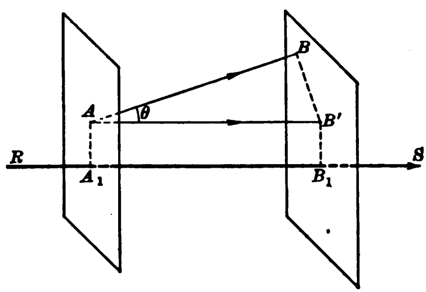

FIG. 61*c*.

that is, *the projection of a segment AB on a directed line RS is equal to the product of the length of the segment by the cosine of the angle between the segment and line.*

Projection of a Broken Line. — Let $ABCD$ be a broken line joining A and D. Project on RS. If segments of RS are considered opposite in sign when drawn in opposite directions

$$A_1B_1 + B_1C_1 + C_1D_1 = A_1D_1.$$

This is true whether A, B, C, D are in a plane or not. Therefore, *the algebraic sum of the projections of the parts of a broken line joining two points is equal to the projection of the segment from the first to the last of those points.*

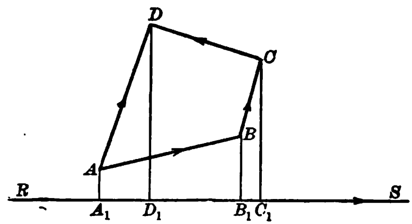

FIG. 61*d*.

Projections on the Axes. — Let P_1P_2 be the segment from P_1 (x_1, y_1, z_1) to P_2 (x_2, y_2, z_2) (Fig. 61*e*). Through P_1 and P_2 pass planes perpendicular to the x-axis meeting it in M_1 and M_2. P_1M_1 and P_2M_2 are then perpendicular to OX. If segments are consid-

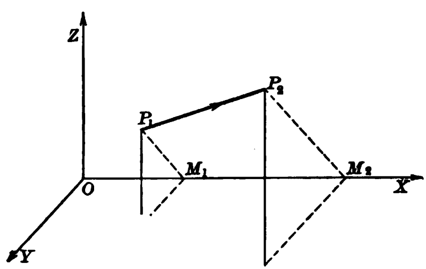

FIG. 61*e*.

ered positive when drawn in the positive direction along the x-axis, the projection of P_1P_2 is

$$M_1M_2 = OM_2 - OM_1 = x_2 - x_1.$$

Similarly the projections on the y- and z-axes are $y_2 - y_1$ and $z_2 - z_1$. Therefore *in length and sign the projection of a segment on a coördinate axis is equal to the difference obtained by subtracting the coördinate of the beginning from that of the end of the segment.*

Exercises

1. Plot the points (0, 0, 1), (1, 1, 0), (1, 1, 1), (−1, 2, 3), (−1, −2, −3).

2. What is the x-coördinate of a point in the yz-plane? What are the x- and y-coördinates of a point on the z-axis?

3. Where are all the points for which $z = -1$? What is the locus of points for which $x = 1$ and $y = 2$?

4. Determine the distance of (x, y, z) from each coördinate axis. What is its distance from the origin?

5. Find the feet of the perpendiculars from (1, 2, 3) to the coördinate planes and to the coördinate axes.

6. Given P (1, 0, 1), Q (2, 1, 5), R (3, −1, 2), find the projections of PQ, QR and RP on the coördinate axes. Show that the sum of the projections on each axis is zero.

7. In exercise 6 find the angles between PQ and the coördinate axes.

8. The projections of AB on OX, OY and OZ are 3, −1 and 2 respectively. Those of BC are 2, −3 and 1. Find the projections of AC.

9. Find the coördinates of the middle point of the segment joining (3, 2, −1) and (2, −3, 4).

10. Given A (2, 1, −2), B (1, 3, 1), C (−1, 7, 7), show that the projections of A, B, C on the xy-plane are three points on a line. Also show that the projections on the yz-plane are points of a line. Hence show that A, B, C lie on a line.

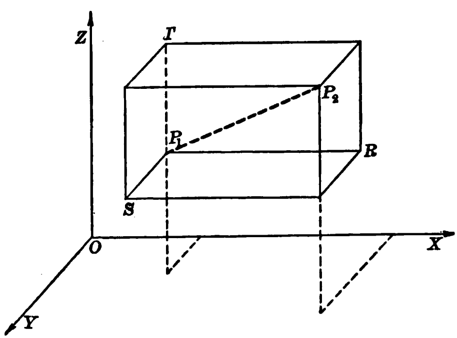

FIG. 62.

Art. 62. Distance between Two Points

Let P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2) be the points. On P_1P_2 as diagonal construct a box with edges parallel to the coördinate axes (Fig. 62). Since projections on parallel lines are equal, the edges

of the box are equal to the projections of P_1P_2 on the coördinate axes. Consequently

$$P_1R = x_2 - x_1, \quad P_1S = y_2 - y_1, \quad P_1T = z_2 - z_1.$$

Since the square on the diagonal of a rectangular parallelopiped is equal to the sum of the squares of its three edges,

$$P_1P_2^2 = P_1R^2 + P_1S^2 + P_1T^2,$$

whence

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \qquad (62)$$

Art. 63. Vectors

A vector is a segment with given length and direction. The components of a vector in space are its projections on the coördinate axes. In this book the vector with components a, b, c will be represented by the symbol $[a, b, c]$. The vector P_1P_2 from $P_1 (x_1, y_1, z_1)$ to $P_2 (x_2, y_2, z_2)$ has components equal to $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$. This is expressed by the equation

$$P_1P_2 = [x_2 - x_1, y_2 - y_1, z_2 - z_1]. \qquad (63)$$

As in plane geometry, the sum of two vectors is obtained by placing the second on the end of the first and drawing the vector from the beginning of the first to the end of the second. If v_1 and v_2 are vectors beginning at the same point, $v_2 - v_1$ is the vector from the end of v_1 to the end of v_2. If r is a number, rv is the vector r times as long as v and having the same direction if r is positive but the opposite direction if r is negative.

The components of the sum of two vectors are obtained by adding corresponding components, those of the difference by subtracting corresponding components and those of the product rv by multiplying the components of v by r.

Example. Find the point $P (x, y, z)$ on the line through $A (-1, 2, 3)$ and $B (3, -4, 2)$ such that $AP = 3\,BP$.

Using the given coördinates,

$$AP = [x + 1, y - 2, z - 3], \quad BP = [x - 3, y + 4, z - 2].$$

If then $AP = 3\,BP$,

$$x + 1 = 3(x - 3), \quad y - 2 = 3(y + 4), \quad z - 3 = 3(z - 2).$$

Consequently $x = 5$, $y = -7$, $z = \frac{3}{2}$.

Exercises

1. Show that the triangle formed by the points (1, 0, 2), (0, −2, 3) and (2, −3, 0) is isosceles.

2. By showing that the sum of the distances AB and BC is equal to AC show that the points A (1, 2, −1), B (0, 5, 1), C (−2, 11, 5) lie on a line.

3. Find the center of the sphere through the points (0, 0, 0), (2, 0, 0), (0, 4, 0) and (0, 0, 6).

4. Given A (1, −1, 2), B (3, 2, −1), find the point on AB produced which is four times as far from A as from B. Also find the point one-fourth of the way from A to B.

5. Given A (1, 3, 1), B (3, 5, 4), find the vectors which are the projections of AB on the coördinate planes. Show that the sum of the squares of the projections is twice the square of AB.

6. Show that the points A (1, 0, −1), B (−2, 1, 3), C (−1, 3, 6), D (2, 2, 2) are the vertices of a parallelogram.

7. AE is the diagonal of a parallelopiped with edges AB, AC and AD. Given A (1, 1, 0), B (2, 3, 0), C (3, 0, 1), D (2, 1, 4), find the coördinates of E.

8. Given A (2, 4, 5), B (1, 3, 0), C (3, 0, 2), D (6, 1, 9), find the middle points of AB and CD. Then find the middle point P of the two middle points. Show that P satisfies the vector equation

$$PA + PB + PC + PD = 0.$$

9. If r and s are numbers the vector $rAB + sAC$ lies in the plane ABC. Consequently, if

$$AD = rAB + sAC,$$

the points A, B, C, D lie in a plane. By finding values of r and s satisfying this equation show that A (0, 0, 1), B (1, −1, 4), C (−2, 1, −3), D (3, 2, 0) lie in a plane.

Art. 64. Direction of a Line

The angles between a directed line and OX, OY and OZ are represented by the letters α, β and γ. These are sometimes called the *direction angles* of the line. The cosines of α, β and γ are called the *direction cosines* of the line.

Let P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2) be two points on the line. Construct a box with P_1P_2 as diagonal and edges parallel to the coördinate axes. If the positive direction along each edge is taken as that of the parallel axis, the direction angles for P_1P_2 are those between P_1P_2 and the edges of the box. Consequently

$$\left.\begin{aligned}\cos\alpha &= \frac{P_1A}{P_1P_2} = \frac{x_2 - x_1}{P_1P_2},\\ \cos\beta &= \frac{P_1B}{P_1P_2} = \frac{y_2 - y_1}{P_1P_2},\\ \cos\gamma &= \frac{P_1C}{P_1P_2} = \frac{z_2 - z_1}{P_1P_2}.\end{aligned}\right\}\tag{64a}$$

The direction cosines of the lines are therefore the components of P_1P_2 *divided by its length.*

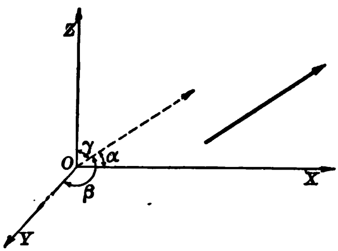

FIG. 64a.

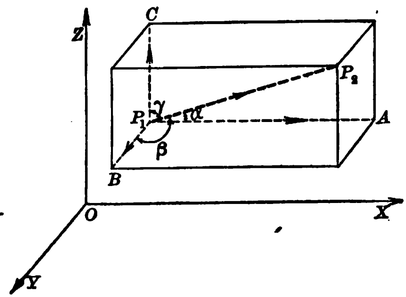

FIG. 64b.

Since $P_1P_2^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$, it is seen that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1, \tag{64b}$$

that is, *the sum of the squares of the direction cosines of a line is equal to unity.*

If it is known that the direction cosines of a line are proportional to three numbers a, b, c, since the sum of the squares of the cosines is equal to unity, the exact cosines are obtained by dividing these proportional numbers by the square root of the sum of their squares. The two square roots give two sets of direction cosines corresponding to the two directions along the line.

Example 1. Find the direction cosines of the vector [2, −3, 5].

The cosines are the ratios of the components to the length of the vector. Consequently,

$$\cos\alpha = \frac{2}{\sqrt{38}}, \quad \cos\beta = \frac{-3}{\sqrt{38}}, \quad \cos\gamma = \frac{5}{\sqrt{38}}.$$

Ex. 2. The direction cosines of a line are proportional to -1, 2 and 1. Find their values and construct a line having the given cosines.

The segment from the origin to the point $(-1, 2, 1)$ has direction cosines proportional to its components -1, 2 and 1. Therefore the line through the origin and the point $(-1, 2, 1)$ has the direction required. Its direction cosines are

$$\cos \alpha = \frac{-1}{\pm\sqrt{6}}, \quad \cos \beta = \frac{2}{\pm\sqrt{6}}, \quad \cos \gamma = \frac{1}{\pm\sqrt{6}},$$

the two signs in the denominators corresponding to the two directions along the line.

Art. 65. The Angle between Two Directed Lines

Let the lines have direction angles $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$. Let OP_1 and OP_2 be lines through the origin with the same directions. Projecting on OP_2,

$$\text{proj. } OP_1 = \text{proj. } OM + \text{proj. } MN + \text{proj. } NP_1.$$

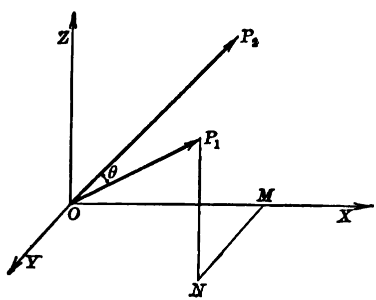

FIG. 65.

Consequently,

$$OP_1 \cos\theta = OM \cos\alpha_2 + MN \cos\beta_2 + NP_1 \cos\gamma_2.$$

But $OM = OP_1 \cos\alpha_1$, $MN = OP_1 \cos\beta_1$, $NP_1 = OP_1 \cos\gamma_1$. Substituting these values and cancelling OP_1,

$$\cos\theta = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2. \qquad (65a)$$

That is, *the cosine of the angle between two directed lines is equal to the sum of the products of corresponding direction cosines.*

If the lines are perpendicular the angle θ is $90°$ and its cosine is zero. In that case

$$\cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2 = 0. \qquad (65b)$$

Therefore, *two lines are perpendicular when the sum of the products of corresponding direction cosines is zero.*

In equation (65*b*) the direction cosines of either line can be replaced by any numbers proportional to them. In particular, *two vectors are perpendicular when the sum of the products of corresponding components is zero.*

Example 1. Find the angle between the vectors $v_1\,[-2,\ 0,\ 1]$ and $v_2\,[1,\ 2,\ 0]$. The direction cosines of the vectors are

$$\cos\alpha_1 = \frac{-2}{\sqrt{5}}, \quad \cos\beta_1 = 0, \quad \cos\gamma_1 = \frac{1}{\sqrt{5}},$$

$$\cos\alpha_2 = \frac{1}{\sqrt{5}}, \quad \cos\beta_2 = \frac{2}{\sqrt{5}}, \quad \cos\gamma_2 = 0.$$

If θ is the angle between the vectors,

$$\cos\theta = -\tfrac{2}{5} + 0 + 0 = -\tfrac{2}{5}.$$

The negative sign shows that the angle is obtuse.

Ex. 2. Show that the points $A\ (4, 2, -1)$, $B\ (3, -2, 3)$, $C\ (1, 1, 0)$ are vertices of a right triangle.

In this case

$$CA = [3,\ 1,\ -1], \quad CB = [2,\ -3,\ 3].$$

The sum of the products of corresponding components is

$$6 - 3 - 3 = 0.$$

The sides CA and CB are then perpendicular and the triangle is a right triangle.

Exercises

1. Determine the direction cosines of the coördinate axes.

2. A straight line in the xy-plane has a slope equal to $\sqrt{3}$. Find its direction angles and direction cosines.

3. Determine the direction cosines of the segment from the origin to the point (2, 3, 6).

4. How many lines through the origin make angles of 60° with the x- and y-axes? What angles do they make with the z-axis?

5. A line makes angles of 45° with the yz- and zx-planes. Find its direction cosines.

6. A line makes equal angles with the coördinate axes. Find those angles.

7. Construct a line with direction cosines proportional to 1, 4 and 8. Find its direction cosines.

8. Find the angles between the vector [1, 2, 2] and the coördinate axes.

9. Determine the angles of the triangle with vertices (1, 0, 0), (0, 1, 0) and (0, 0, 2).

10. Show that the line joining the origin to the point (1, 1, 1) is perpendicular to the line through (1, 1, 0) and (0, 0, 2).

11. A line L_1 in the xz-plane makes an angle of $60°$ with the line L_2 in the xy-plane with equation $x + 2y = 4$. Find the angle between L_1 and the x-axis.

12. Find the angles between the segments from (1, 1, 1) to the points (2, 0, 3), (0, 3, -2) and (3, 1, 3). Show that one of these angles is equal to the sum of the other two. What do you conclude about the four points?

13. Show that the line through the origin and the point (1, -1, -1) is perpendicular to the line joining any pair of the four points in Ex. 12.

Art. 66. Cylindrical and Spherical Coördinates

Let O be the origin and OX the initial line of a system of polar coördinates in the xy-plane. Let the projection of a point P on the xy-plane have coördinates r and θ. The *cylindrical coördinates* of P are r, θ and z. The angle θ is considered positive when measured from OX toward OY and r is positive when N lies on the terminal side of the angle.

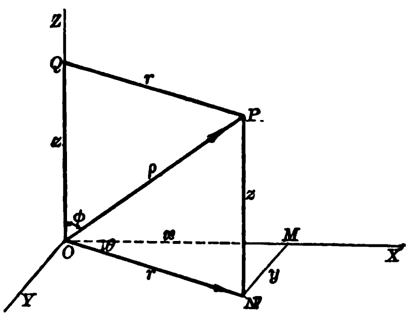

Fig. 66.

In the plane ONP let ρ and ϕ be the polar coördinates of P, the z-axis being initial line and O the origin. The *spherical coördinates* of P are ρ, θ and ϕ. In this case ϕ is considered positive when measured from OZ toward the terminal side of θ and ρ is positive when P lies on the terminal side of ϕ. In some books these coördinates are called polar.

The relations of the rectangular, cylindrical and spherical coördinates of a point can easily be determined from the right triangles

OMN and *OPQ* in Fig. 66. The most important equations connecting the coördinates are

$$\left.\begin{array}{ll} x = r\cos\theta, & y = r\sin\theta, \\ r = \rho\sin\phi, & z = \rho\cos\phi, \\ r^2 = x^2 + y^2, & \rho^2 = x^2 + y^2 + z^2. \end{array}\right\} \tag{66}$$

Example. Determine the cylindrical and spherical coördinates of the point (1, 2, 3).

From Fig. 66 it is seen that

$$\tan\theta = \frac{y}{x} = 2, \qquad r = \sqrt{x^2 + y^2} = \sqrt{5},$$

$$\tan\phi = \frac{r}{z} = \frac{1}{3}\sqrt{5}, \quad \rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{14}.$$

The cylindrical coördinates of the point are then

$$r = \sqrt{5}, \quad \theta = \tan^{-1} 2, \qquad z = 3$$

and its spherical coördinates are

$$\rho = \sqrt{14}, \quad \theta = \tan^{-1} 2, \qquad \phi = \tan^{-1}\left(\tfrac{1}{3}\sqrt{5}\right).$$

Exercises

1. Using cylindrical coördinates (r, θ, z) construct the points $\left(2, \frac{\pi}{2}, 1\right)$, $\left(-3, \frac{\pi}{4}, 0\right)$, $\left(2, -\frac{\pi}{6}, -1\right)$ and find their rectangular and spherical coördinates.

2. Using spherical coördinates (ρ, θ, ϕ) construct the points $\left(1, \frac{\pi}{4}, \frac{\pi}{3}\right)$, $\left(-1, \frac{\pi}{6}, 0\right)$, $\left(2, \pi, \frac{\pi}{2}\right)$ and find their rectangular and cylindrical coördinates.

3. What is the locus of all points for which $r = 1$? For which $\theta = \frac{\pi}{6}$? What is the locus of points for which $r = 1$ and $\theta = \frac{\pi}{6}$?

4. Determine the locus of points in each of the following cases:

$$\text{(a) } \rho = 2, \text{ (b) } \theta = \frac{2}{3}\pi, \text{ (c) } \phi = \frac{\pi}{6}, \text{ (d) } \phi = \frac{\pi}{2}, \text{ (e) } \phi = \pi.$$

5. Determine the locus of points in each of the following cases:

(a) $r = 2,\ z = 3$, (b) $\theta = 60°,\ z = -2$, (c) $r = -1,\ \theta = \pi$.

6. Determine the locus of points in each of the following cases:

$$\text{(a) } \rho = 2,\ \theta = \pi, \quad \text{(b) } \rho = -2,\ \phi = \frac{\pi}{6}, \quad \text{(c) } \theta = \frac{\pi}{3},\ \phi = \frac{\pi}{4}.$$

7. Determine the distances of a point from the coördinate axes in spherical and cylindrical coördinates.

8. The spherical coördinates of P are $\rho = 2$, $\theta = 30°$, $\phi = 45°$. Find the angles between OP and the coördinate axes.

CHAPTER 10

SURFACES

Art. 67. Loci

An equation represents a locus if every point on the locus has coördinates satisfying the equation and every point with coördinates satisfying the equation lies on the locus.

One equation between the coördinates of a point in space usually represents a surface. Thus, the equation $z = 0$ represents the xy-plane, for any point in the xy-plane has a z-coördinate equal to zero and every point with z-coördinate equal to zero lies in the xy-plane. Similarly the equation $x^2 + y^2 + z^2 = 1$ represents a sphere with radius 1 and center at the origin. In particular cases one equation may represent a straight line or curve. Thus, the only real points for which $x^2 + y^2 = 0$ are the points $x = 0, y = 0$ on the z-axis.

Two simultaneous equations usually represent a curve or straight line; for each equation represents a surface and the two equations represent the intersection of two surfaces, that is, a straight line or curve. Thus, the equations

$$x^2 + y^2 + z^2 = 3, \quad z = 1$$

represent the circle in which a sphere and plane intersect.

Three simultaneous equations are usually satisfied by the coördinates of a definite number of points. These points are found by solving the equations simultaneously. In particular cases the equations may have no solution or may be satisfied by the coördinates of all points on a curve or surface.

Art. 68. Equation of a Plane

A line perpendicular to a curve or surface is called a *normal* to that curve or surface.

Let a normal DN to a plane (Fig. 68a) have direction angles α, β, γ. Let P_1 (x_1, y_1, z_1) be a fixed point and P (x, y, z) a variable point in the plane. The direction cosines of DN are $\cos\alpha$, $\cos\beta$ and $\cos\gamma$. Those of P_1P are proportional to $x - x_1$, $y - y_1$, $z - z_1$. Since DN is perpendicular to the plane it is perpendicular to P_1P. Therefore, by Art. 65,

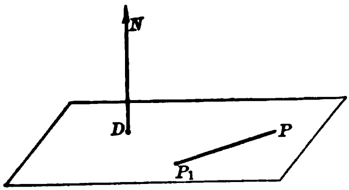

FIG. 68a.

$$(x - x_1)\cos\alpha + (y - y_1)\cos\beta + (z - z_1)\cos\gamma = 0. \qquad (68a)$$

This is the equation of the plane through (x_1, y_1, z_1) whose normal makes angles α, β, γ with the coördinate axes.

Let the direction cosines of the normal be proportional to A, B, C. Then

$$\cos\alpha : \cos\beta : \cos\gamma = A : B : C.$$

Since the cosines in equation (68a) can be replaced by any proportional numbers, that equation is equivalent to

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0, \qquad (68b)$$

which is therefore the equation of the plane through (x_1, y_1, z_1) perpendicular to the line with direction cosines proportional to A, B, C.

First Degree Equation. — Equations (68a) and (68b) are of the first degree in x, y, z. Therefore *any plane has an equation of the first degree in rectangular coördinates.*

Conversely, *any equation of the first degree in rectangular coördinates represents a plane;* for any such equation has the form

$$Ax + By + Cz + D = 0, \qquad (68c)$$

A, B, C, D being constant. Let x_1, y_1, z_1 satisfy this equation. Then

$$Ax_1 + By_1 + Cz_1 + D = 0.$$

Subtraction gives

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0,$$

which is equation (68b). Therefore *any equation of the form* (68c)

represents a plane whose normal has direction cosines proportional to A, B, C.

Example 1. Construct the plane with equation $2x + 3y + z = 6$.

The plane can be determined by its intersections with the coördinate axes. Where the plane crosses the x-axis y and z are zero and so $x = 3$. Similarly it crosses the y-axis at $y = 2$ and the z-axis at $z = 6$. The *intercepts* on the three axes are 3, 2 and 6.

Ex. 2. Construct the plane represented by the equation $x - 2y + z = 0$.

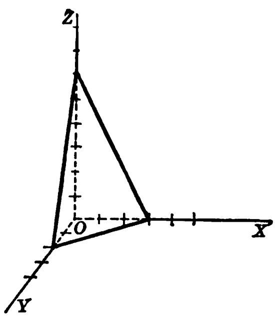

FIG. 68*b*.

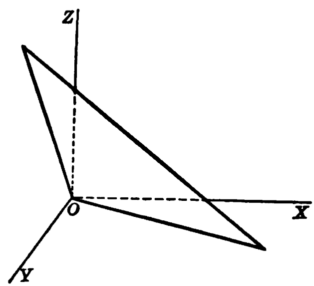

FIG. 68*c*.

The plane passes through the origin and so its intercepts are all zero. It can be determined by its intersections with the coördinate planes. It cuts the xy-plane in the line $z = 0$, $x = 2y$ and the yz-plane in the line $x = 0$, $z = 2y$. In their respective planes these lines are constructed from the equations $x = 2y$ and $z = 2y$ as in plane geometry. The plane through these lines is the one required.

Ex. 3. Find the equation of the plane through $(1, -2, 4)$ perpendicular to the line through A $(2, 1, 0)$ and B $(1, 2, 3)$.

The direction cosines of AB are proportional to $1 - 2$, $2 - 1$, $3 - 0$. Using these values instead of A, B, C in (68*b*) the equation of the plane is found to be

$$-(x - 1) + (y + 2) + 3(z - 4) = 0.$$

Ex. 4. Find the angle between the planes

$$x - y + z = 1, \quad 2x + 3y - z = 2.$$

Between two planes are two angles less than 180°. It is shown in solid geometry that these angles are equal to the angles between lines perpendicular to the two planes. The normals to the two planes have direction cosines proportional to the coefficients 1, −1, 1 and 2, 3, −1. The exact cosines of two normals are then

$$\cos\alpha_1 = \frac{1}{\sqrt{3}}, \quad \cos\beta_1 = \frac{-1}{\sqrt{3}}, \quad \cos\gamma_1 = \frac{1}{\sqrt{3}},$$

$$\cos\alpha_2 = \frac{2}{\sqrt{14}}, \quad \cos\beta_2 = \frac{3}{\sqrt{14}}, \quad \cos\gamma_2 = \frac{-1}{\sqrt{14}}.$$

By equation (65*a*) the angle θ between the two normals satisfies the equation

$$\cos\theta = \frac{-2}{\sqrt{42}} = \frac{-1}{\sqrt{13}}.$$

The negative sign shows that this is the obtuse angle. The acute angle between the two normals or between the two planes is $\cos^{-1}\left(\frac{1}{\sqrt{13}}\right)$.

Exercises

Construct the planes represented by the following equations and find the direction cosines of their normals:

1. $x + 2y + 4z = 4.$
2. $x - y + 3z = 5.$
3. $x + y + z = 0.$
4. $2x - 3y = 0.$
5. $3x + 4z = 0.$
6. $z + 5 = 0.$

7. Find the equation of the plane through the origin perpendicular to the line with direction angles $\alpha = 60°$, $\beta = 45°$, $\gamma = 60°$.

8. Find the equation of the plane through (1, 1, 0) perpendicular to the vector [3, −5, 4].

9. Find the equation of the plane with intercepts on OX, OY and OZ equal to 1, 2 and 3.

10. Show that the planes $x + 2y - z = 1$ and $2x + 4y - 2z = 3$ are parallel.

11. Show that the planes $x + y - z = 0$ and $2x - 3y - z = 0$ are perpendicular.

12. Find the angle between the planes $x + 2y + 2z = 0$ and $x - 4y + 8z = 9$.

13. Show that the angle between a line and a plane is the complement of the angle between the line and the normal to the plane. Find the angle between the plane $x - 2y - z = 0$ and the line through the points (3, 0, 1) and (0, 2, −1).

Art. 69. Equation of a Sphere

A sphere is the locus of points at a constant distance from a fixed point. The fixed point is the center, and the constant distance the radius of the sphere.

Let $C(x_1, y_1, z_1)$ be the center of a sphere with radius r. If $P(x, y, z)$ is any point on the sphere, its equation is

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2. \qquad (69a)$$

When expanded the equation of the sphere has the form

$$x^2 + y^2 + z^2 + Ax + By + Cz + D = 0. \qquad (69b)$$

Conversely, any equation of this form represents a sphere if it represents a real surface. To show this, complete the squares in x, y and z separately. The result will have the form

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = d.$$

If d is positive this represents a sphere with center (a, b, c) and radius $\sqrt{d}$. If d is zero the locus is the single point (a, b, c). If d is negative there is no real locus. Hence, whenever the equation represents a surface, that surface is a sphere.

Example. Determine the center and radius of the sphere

$$x^2 + y^2 + z^2 - 2x + 3y = 0.$$

Completing the squares,

$$(x - 1)^2 + (y + \tfrac{3}{2})^2 + z^2 = \tfrac{13}{4}.$$

The center is $(1, -\frac{3}{2}, 0)$ and the radius is $\frac{1}{2}\sqrt{13}$.

Art. 70. Equation of a Cylindrical Surface

A *cylindrical surface* is one generated by lines parallel to a fixed line and cutting a fixed curve. The lines are called *generators* and the fixed curve is called a *directrix*.

A cylindrical surface with generators parallel to a coördinate axis is represented in rectangular coördinates by an equation containing

only two coördinates. To show this let the generators be parallel to the z-axis. Let the surface intersect the xy-plane in the curve $z = 0, f(x, y) = 0$. Any point $P(x, y, z)$ on the surface has the same x and y as its projection $Q(x, y, 0)$ on the xy-plane. Since the coördinates x and y of Q satisfy the equation $f(x, y) = 0$, so do the coördinates x, y, z of P. Also, if the coördinates of any point P satisfy the equation $f(x, y) = 0$, P is in the vertical line through a point Q of the curve and so lies on the surface. Therefore $f(x, y) = 0$ is the equation of the cylindrical surface.

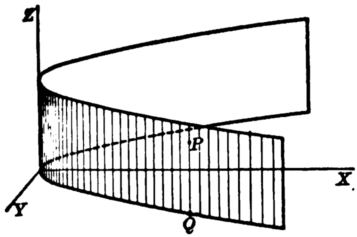

FIG. 70.

Conversely, *any equation in two rectangular coördinates represents a cylindrical surface with generators parallel to the axis of the missing coördinate.* To show this let the equation be $f(x, y) = 0$. Any point P with coördinates satisfying this equation lies in a vertical line through a point Q of the curve $z = 0, f(x, y) = 0$, and any point in such a vertical line has coördinates satisfying the equation. Therefore the equation $f(x, y) = 0$ represents a cylindrical surface whose directrix is the curve $z = 0, f(x, y) = 0$ in the xy-plane.

Example. Find the equation of the cylindrical surface with generators parallel to the x-axis cutting the yz-plane in the circle with radius 2 and center (0, 1, 1).

In the yz-plane the equation of the circle is

$$(y - 1)^2 + (z - 1)^2 = 4.$$

This considered as an equation in space represents the given cylindrical surface.

Art. 71. Surface of Revolution

The surface described by a plane curve revolving about an axis in its plane is called a *surface of revolution.*

In the xz-plane let $f(x, z) = 0$ be the equation of a curve. Let

this curve be revolved about the z-axis. The cylindrical coördinates r, z of any point P on the resulting surface are equal to the rectangular coördinates x, z of a point Q on the curve in the xz-plane. Since the coördinates of Q satisfy the equation $f(x, z) = 0$, the cylindrical coördinates of P satisfy the equation

$$f(r, z) = 0.$$

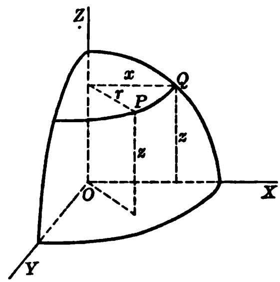

FIG. 71.

This is then the equation of the surface described by revolving the curve $y = 0, f(x, z) = 0$ about the z-axis.

Since $r = \sqrt{x^2 + y^2}$, the rectangular equation of the surface is

$$f(\sqrt{x^2 + y^2}, z) = 0.$$

Similar equations are found for the surfaces obtained by revolving about the x- or y-axis. Therefore, *to find the rectangular equation of the surface described by rotating a curve in a coördinate plane about a coördinate axis in that plane, leave the coördinate corresponding to the axis of rotation unchanged in the plane equation of the curve and replace the other coördinate by the square root of the sum of the squares of the other two.*

Example 1. Find the equation of the surface described by rotating the parabola $y^2 = 2x$ about the x-axis.

This means that the parabola in the xy-plane with plane equation $y^2 = 2x$ is to be rotated about the x-axis. The equation of the resulting surface is obtained from $y^2 = 2x$ by leaving x unchanged and replacing y by $\sqrt{y^2 + z^2}$. The required equation is then

$$y^2 + z^2 = 2x.$$

Ex. 2. Show that

$$(x^2 + y^2)^2 + z^2(x^2 + y^2) = 1$$

is the equation of a surface of revolution.

This is indicated by the fact that the equation contains x and y only in the combination $x^2 + y^2$. The surface is generated by

revolving about the z-axis the curve with plane equation $x^4 + z^2x^2 = 1$.

Exercises

Describe the surfaces represented by the following equations:

1. $x^2 + y^2 + z^2 = 9$.
2. $x^2 + y^2 + z^2 = 4x$.
3. $x^2 + y^2 + z^2 - 3x + 2y + z = 0$.
4. $2x^2 + 2y^2 + 2z^2 - 5x + 8y - 6z + 4 = 0$.
5. $x^2 + y^2 = 2x$.
6. $x^2 - y^2 = a^2$.
7. $y^2 = az$.
8. $z = \sin x$.
9. $x^2 + y^2 = 2z$.
10. $x^2 + y^2 = z^2$.
11. $x^2 - y^2 = z^2$.
12. $x^2 - z^2 = 0$.
13. $r = a \cos \theta$.
14. $r = az + b$.
15. $\rho = a \cos \phi$.
16. $\rho \sin \phi = a$.

Find the rectangular, cylindrical and spherical equations of the following surfaces:

17. Sphere with radius a and center at the origin.

18. Sphere with center $(0, 0, a)$ passing through the origin.

19. Right circular cylinder with axis OZ and radius a.

20. Right circular cone with axis OZ and vertical angle $90°$ (between generators in a plane through the axis).

Find the rectangular equations of the following surfaces:

21. Right circular cylinder tangent to the xy-plane along the x-axis.

22. Parabolic cylinder with generators parallel to OY and directrix a parabola in the xz-plane with axis OZ and vertex at the origin.

23. Elliptical cylinder with generators parallel to the z-axis and directrix an ellipse with axes along OX and OY.

24. Prolate spheroid generated by revolving an ellipse about its major axis.

25. Hyperboloid of revolution generated by revolving a hyperbola about one of its axes.

26. Paraboloid of revolution generated by revolving a parabola about its axis.

27. Taurus generated by revolving a circle about a line in its plane not cutting the circle.

Art. 72. Graph of an Equation

To construct the graph of a given equation it is customary to draw a series of plane sections and from these sections to determine the appearance of the surface. The sections generally used are those in and parallel to the coördinate planes.

Example 1. The ellipsoid,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The sections of this surface in the xz- and yz-planes are the ellipses

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 \quad \text{and} \quad \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

having a common axis on $Z'Z$ (Fig. 72*a*). The section in a horizontal plane $z = k$ is an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2},$$

with axes parallel to OX and OY. Since the axes of this ellipse are in the xz- and yz-planes and end on the surface, they are chords of the ellipses in those planes. The surface, called an ellipsoid, is thus generated by horizontal ellipses whose axes are chords of the two vertical ellipses.

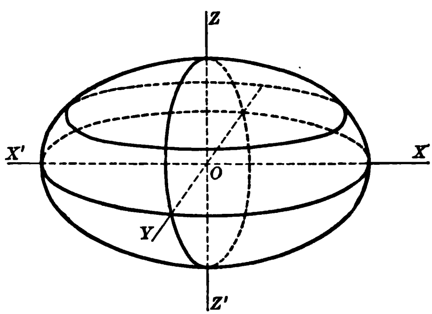

FIG. 72*a*.

The quantities a, b, c, called the semi-axes, are equal to the intercepts on the coördinate axes. If two of these semi-axes are equal the ellipsoid is called a spheroid. It is then a surface of revolution obtained by revolving an ellipse about one of its axes. If the semi-axes are all equal the ellipsoid is a sphere.

Ex. 2. **The hyperboloid of one sheet,**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

The sections in the xz- and yz-planes are hyperbolas

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \quad \text{and} \quad \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

with a common axis $Z'Z$ (Fig. 72*b*). The section in a horizontal plane $z = k$ is an ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}$$

with the ends of its axes on the two hyperbolas.

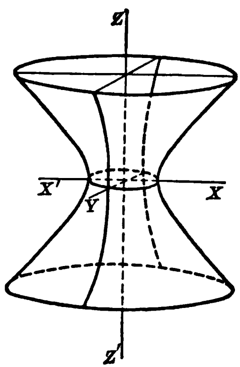

FIG. 72*b*.

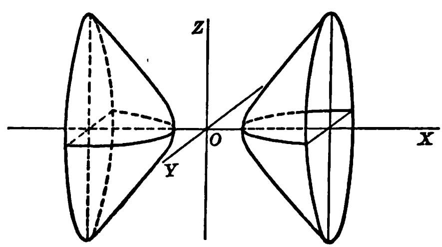

FIG. 72*c*.

If $a = b$ the surface is a hyperboloid of revolution obtained by revolving a hyperbola about its conjugate axis.

Ex. 3. **The hyperboloid of two sheets,**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

The section in a plane $x = k$ is imaginary if $|x| < a$. The surface therefore consists of two parts, one on the right of $x = a$ and one on the left of $x = -a$. The surface cuts the xy- and xz-planes in hyperbolas and is generated by ellipses parallel to the yz-plane whose axes are chords of these hyperbolas (Fig. 72*c*).

If $b = c$ the surface is a hyperboloid of revolution obtained by revolving a hyperbola about its transverse axis.

Ex. 4. **The elliptic paraboloid,**

$$z = ax^2 + by^2,$$

a and b having the same algebraic sign.

Sections in the xz- and yz-planes are parabolas

$$z = ax^2 \text{ and } z = by^2.$$

The section in a plane $z = k$ is an ellipse

$$ax^2 + by^2 = k$$

with axes parallel to OX and OY if a, b and k have the same sign and imaginary if k is opposite in sign from a and b. The surface is thus generated by horizontal ellipses whose axes are chords of the two parabolas.

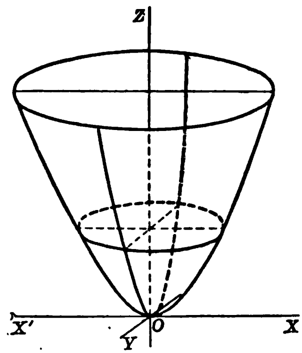

FIG. 72d.

If $a = b$ the surface is a paraboloid of revolution obtained by revolving a parabola about its axis.

Ex. 5. **The hyperbolic paraboloid,**

$$z = ax^2 - by^2,$$

a and b being positive.

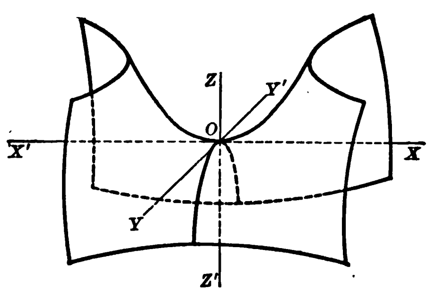

FIG. 72e.

The section in the xz-plane is a parabola $z = ax^2$ extending upward. The section in the yz-plane is a parabola $z = -by^2$ ex-

tending downward. The section in the xy-plane is a pair of lines $ax^2 - by^2 = 0$. The section in a horizontal plane $z = k$ is a hyperbola $ax^2 - by^2 = k$ whose transverse axis is a chord of the parabola (in the xz- or yz-plane) cut by that plane. The surface has the general shape of a saddle.

Ex. 6. **The hyperbolic paraboloid,**

$$z = kxy.$$

This is a saddle-shaped surface similar to that in Ex. 5. The hyperbolas in horizontal planes are however rectangular with asymptotes parallel to the x- and y-axes.

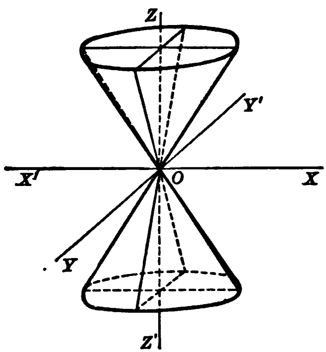

FIG. 72*f*.

Ex. 7. **The cone,**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}.$$

The sections in the xz- and yz-planes are pairs of lines

$$\frac{x}{a} = \pm\frac{z}{c}, \quad \text{and} \quad \frac{y}{b} = \pm\frac{z}{c}.$$

The surface is generated by horizontal ellipses the ends of whose axes are on these lines.

Ex. 8. Describe the surface with spherical equation

$$\rho = a \sin\phi.$$

Since θ does not occur in the equation the surface is one of revolution about the z-axis. The section in the xz-plane is a circle tangent to the z-axis at the origin. The graph is a doughnut-shaped surface generated by revolving this circle about the z-axis.

Exercises

Draw the graphs and describe the surfaces represented by the following equations:

1. $x^2 + 2y^2 + 3z^2 = 6$.
2. $(x-1)^2 + 2(y-2)^2 + 3(z-3)^2 = 6$.
3. $x^2 + 4(y^2 + z^2) = 12$.
4. $x^2 - y^2 + z^2 = 1$.
5. $x^2 - y^2 + z^2 - 2x + 4y = 4$.
6. $x^2 - y^2 - z^2 = 3$.
7. $x^2 + z^2 = y$.
8. $y^2 + 2z^2 - 4x^2 = 12$.
9. $x = z^2 + 2y^2 + 2z - 12y + 19$

10. $x = 2\,yz.$

11. $x^2 - y^2 = z^2.$

12. $(x+2)^2 + 4\,(y-1)^2 = (z-3)^2.$

13. $y^2 = z^3.$

14. $z = xy + x + y.$

15. $r = a \cos 2\,\theta.$

16. $\rho = a \cos \theta.$

17. $\rho \cos \phi = 2.$

18. $r^2 + z^2 = a^2.$

19. $rz = k.$

20. $xyz = a^3.$

CHAPTER 11

LINES AND CURVES

Art. 73. The Straight Line

A straight line is the intersection of two planes. It is then represented in rectangular coördinates by two first degree equations. The line is best constructed by finding two points on it. The most convenient points for this purpose are usually its intersections with two coördinate planes. If the line passes through the origin a second point can be found by assigning an arbitrary value to one of the coördinates and calculating the values of the other two.

Example 1. Construct the line represented by the equations

$$y + z = 3, \quad 4x + 3y - 3z = 3,$$

and find its direction cosines.

The line cuts the yz-plane where $x = 0$, that is, where

$$y + z = 3, \quad 3y - 3z = 3.$$

The solution of these equations is $y = 2$, $z = 1$. The intersection with the yz-plane is then A (0, 2, 1). In the same way the intersection with the xz-plane is found to be B (3, 0, 3). Draw the line through A and B (Fig. 73a). Its direction cosines are

$$\cos\alpha = \frac{3}{\sqrt{17}}, \quad \cos\beta = \frac{-2}{\sqrt{17}}, \quad \cos\gamma = \frac{2}{\sqrt{17}}.$$

Line through a Point with a Given Direction. — Let the line pass through P_1 (x_1, y_1, z_1) and have direction angles α, β, γ (Fig. 73b). If P (x, y, z) is any point on the line

$$x - x_1 = P_1P\cos\alpha, \quad y - y_1 = P_1P\cos\beta, \quad z - z_1 = P_1P\cos\gamma.$$

Hence

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma}. \tag{73a}$$

These are the equations of the line. Since two quantities equal to a third are equal to each other, they are equivalent to two independent equations.

If the cosines are proportional to A, B, C, equations (73a) are equivalent to

$$\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C}. \tag{73b}$$

These are the equations of a line through (x_1, y_1, z_1) *with direction cosines proportional to* A, B, C.

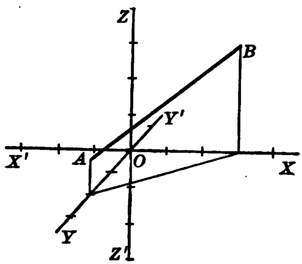

FIG. 73a.

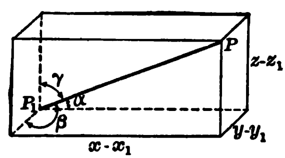

FIG. 73b.

Ex. 2. Find the equations of the line through (1, 0, 1) and (−2, 1, 0).

The direction cosines of the line are proportional to 3, −1, 1. Since the line passes through (1, 0, 1) its equations are then

$$\frac{x - 1}{3} = \frac{y - 0}{-1} = \frac{z - 1}{1}.$$

These are equivalent to the two equations

$$\frac{x - 1}{3} = \frac{y - 0}{-1}, \qquad \frac{y - 0}{-1} = \frac{z - 1}{1},$$

and consequently to $x + 3y = 1$, $y + z = 1$.

Exercises

Construct the lines represented by the following equations and find their direction cosines:

1. $x + y = 1, \quad y = 2z.$
2. $x - y = z, \quad x + y = 0.$
3. $x + y + z = 1, \quad 2x - 3y + 4z = 5.$
4. $x = 2, \quad y = 3.$

5. Find the equations of the line through the points (2, 3, −1) and (3, 4, 2).

6. Find the equations of the line through (0, 1, 2) parallel to the vector [3, 1, 5].

7. Find the equation of the line through (1, 1, 1) perpendicular to the plane $x + 2y - z = 3$.

8. Find the angle between the lines $x + y - z = 0$, $x + z = 0$ and $x - y = 1$, $x - 3y + z = 0$.

9. Find the angle between the line $x - y + z = 1$, $x = 2$ and the plane $z = x - 3y$.

10. Show that the lines $x + y + z = 1$, $2x - y + 3z = 2$ and $3y - z = 2$, $3x + 4z = 1$ are parallel.

11. Show that the lines $x + 2y = 1$, $2y - z = 1$ and $x - y = 1$, $x - 2y = 3$ meet in a point and are perpendicular.

Art. 74. Curves

A curve is the intersection of two surfaces. It is then represented by two simultaneous equations. Since an indefinite number of surfaces can be passed through a curve, it can be represented in an indefinite number of ways by a pair of equations.

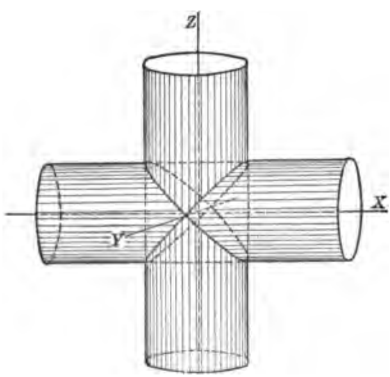

FIG. 74a.

Example 1. Show that the equations

$$x^2 + y^2 + z^2 = 1,$$
$$x + y + z = 1$$

represent a circle.

The first equation represents a sphere, the second a plane. The two equations represent the circle in which the sphere and plane intersect.

Ex. 2. Determine the locus represented by the equations

$$x^2 + y^2 = a^2, \quad y^2 + z^2 = a^2.$$

These equations represent circular cylinders of radius a (Fig. 74a). The locus required is the intersection of the cylinders. Subtraction of the equations gives $x^2 - z^2 = 0$. Therefore $x = \pm z$. All points

of the intersection thus lie in the two planes $x = z$ and $x = -z$. The locus is two ellipses in which the planes cut the cylinders.

Projecting Cylinders. — If a rectangular coördinate is eliminated from the equations of a curve, the resulting equation usually represents the cylindrical surface with the curve as directrix and generators parallel to the axis of the eliminated coördinate. The intersection of this cylinder with the plane of the other two coördinates is then the projection of the curve on that coördinate plane. These statements are illustrated in the examples solved below.

Ex. 3. Find the cylinders with generators parallel to the coordinate axes and cutting the curve

$$z = x^2 + y^2, \quad z = x.$$

Eliminating z we get $x^2 + y^2 = x$. Since this equation contains only x and y, it represents a cylinder with generators parallel to the z-axis. Since the equation of the cylinder is a consequence of the equations of the curve, all points on the curve lie on the cylinder. Furthermore, if x and y satisfy the equation of the cylinder, a value of z can be found such that x, y, z satisfy the equations of the curve. That is, each generator of the cylinder cuts the curve. Therefore $x^2 + y^2 = x$ is the equation of the cylinder generated by lines parallel to the z-axis and cutting the curve. In the same way the equation of

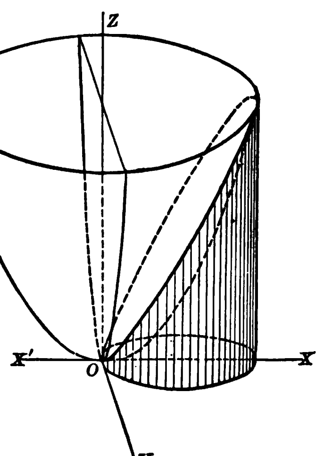

FIG. 74*b*.

the cylinder parallel to the x-axis is found to be $y^2 + z^2 = z$. Lines parallel to the y-axis and cutting the curve lie in the plane $z = x$. They however generate only the strip of this plane in which the curve lies. This shows that the elimination of a coör-

dinate may give more than the surface generated by lines cutting the curve and parallel to the axis of that coördinate.

Ex. 4. Find the equations of the projections of the line

$$2x + y - z = 0, \quad x - y + 2z = 3$$

on the coördinate planes.

Eliminating z, we get

$$5x + y = 3.$$

This is the equation of a plane through the line parallel to the z-axis. The equation of the projection on the xy-plane is then

$$z = 0, \quad 5x + y = 3.$$

In the same way the projections on the xz- and yz-planes are found to be $y = 0$, $3x + z = 3$ and $x = 0$, $3y - 5z + 6 = 0$.

Exercises

Draw the following curves and find their projections on the coördinate planes:

1. $x+y+z=1, \quad x^2+y^2+z^2=1.$
2. $x^2 + y^2 = a^2, \quad x^2 + y^2 = z^2.$
3. $z^2 = x^2 + y^2, \quad x + y = 1.$
4. $z = xy, \quad z = 2x.$
5. $z = x^2 + y^2, \quad z = y^2 + z^2.$
6. $x^2+y^2+z^2=a^2, \quad x^2+z^2=ax.$
7. $r = a, \quad \theta = z.$
8. $\phi = \frac{1}{3}\pi, \quad \rho = \theta.$

9. Show that the equations $x^2 + y^2 - z^2 = 1$, $y - z = 1 - x$ represent a pair of lines.

10. Show that the circle $x^2 + y^2 + z^2 = 6$, $y + 2x = 1$ and the line $y + z = 1$, $x + y + z = 2$ intersect.

11. Find the intersection of the circle $x^2 + y^2 + z^2 + 6x = 0$, $x + y = 0$ and the plane $x + z = 1$.

Art. 75. Parametric Equations

The locus of a point whose coördinates are given functions of a parameter is usually a curve. For, if one of the equations is solved for the parameter and the value substituted in the other two, two equations between the coördinates are obtained. Thus

$$x = t, \quad y = t^2, \quad z = t^3$$

are parametric equations of a curve. To plot the curve, we can assign values to the parameter, calculate the corresponding values of the coördinates and plot the resulting points. By eliminating

the parameter the equation of a surface through the curve is obtained. If the parameter is eliminated between two of the parametric equations, the resulting equation considered as an equation in a coördinate plane represents the projection of the curve on that plane. For example the projections of the curve given above have the equations

$$y = x^2, \quad z = x^3, \quad z^2 = y^3.$$

Example. — **The helix** is a curve traced on the surface of a right circular cylinder by a point that advances in the direction of the axis of the cylinder while it rotates around the axis of the cylinder, the amount of advance being proportional to the angle of rotation.

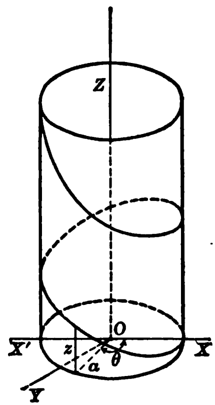

FIG. 75.

To find the equations of the helix, let the axis of z be the axis of the cylinder, a the radius of the cylinder, and let the x-axis pass through a point of the helix. If θ is the angle of rotation,

$$x = a\cos\theta, \quad y = a\sin\theta, \quad z = k\theta,$$

k being the ratio of the advance in the direction of the axis to the angle of rotation.

Exercises

Construct the following curves and find their projections on the coördinate planes:

1. $x = 1 + t$, $y = 2 - t$, $z = 3t$.
2. $x = \cos\theta$, $y = \sin\theta$, $z = 2\theta$.
3. $x = t\sin t$, $y = t\cos t$, $z = t$.
4. $x = \sin t$, $y = \cos t$, $z = \tan t$.
5. A conical helix is described by a point moving on the surface of a right circular cone, the distance of the point from the vertex of the cone being proportional to the angle of rotation about the axis. Find parametric equations for the curve.
6. Find the equation of the twisted surface generated by perpendiculars from points of a helix to the axis of the cylinder on which it lies.
7. Neglecting friction the position of a bullet starting from the origin with velocity $[a, b, c]$ after t seconds is given by the equations

$$x = at, \quad y = bt, \quad z = ct - 16.1\,t^2.$$

Construct the curve and find its projections on the coördinate planes if it starts with a velocity of 1000 ft. per second in the direction $\alpha = 60°$, $\beta = 45°$, $\gamma = 60°$.

8. The wheel of a gyroscope rotates with constant speed around its axis while the axis turns with constant speed about a fixed point of itself. Find equations of the curve described by a fixed point on the periphery of the wheel.

Natural Values of Trigonometric Functions

Deg.	Rad.	Sin.	Cos.	Tan.	Deg.	Rad.	Sin.	Cos.	Tan.
0	.0000	.0000	1.0000	.0000	**45**	.7854	.7071	.7071	1.0000
1	.0175	.0175	.9998	.0175	46	.8029	.7193	.6947	1.0355
2	.0349	.0349	.9994	.0349	47	.8203	.7314	.6820	1.0724
3	.0524	.0523	.9986	.0524	48	.8378	.7431	.6691	1.1106
4	.0698	.0698	.9976	.0699	49	.8552	.7547	.6561	1.1504
5	.0873	.0872	.9962	.0875	**50**	.8727	.7660	.6428	1.1918
6	.1047	.1045	.9945	.1051	51	.8901	.7771	.6293	1.2349
7	.1222	.1219	.9925	.1228	52	.9076	.7880	.6157	1.2799
8	.1396	.1392	.9903	.1405	53	.9250	.7986	.6018	1.3270
9	.1571	.1564	.9877	.1584	54	.9425	.8090	.5878	1.3764
10	.1745	.1736	.9848	.1763	**55**	.9599	.8192	.5736	1.4281
11	.1920	.1908	.9816	.1944	56	.9774	.8290	.5592	1.4826
12	.2094	.2079	.9781	.2126	57	.9948	.8387	.5446	1.5399
13	.2269	.2250	.9744	.2309	58	1.0123	.8480	.5299	1.6003
14	.2443	.2419	.9703	.2493	59	1.0297	.8572	.5150	1.6643
15	.2618	.2588	.9659	.2679	**60**	1.0472	.8660	.5000	1.7321
16	.2793	.2756	.9613	.2867	61	1.0647	.8746	.4848	1.8040
17	.2967	.2924	.9563	.3057	62	1.0821	.8829	.4695	1.8807
18	.3142	.3090	.9511	.3249	63	1.0996	.8910	.4540	1.9626
19	.3316	.3256	.9455	.3443	64	1.1170	.8988	.4384	2.0503
20	.3491	.3420	.9397	.3640	**65**	1.1345	.9063	.4226	2.1445
21	.3665	.3584	.9336	.3839	66	1.1519	.9135	.4067	2.2460
22	.3840	.3746	.9272	.4040	67	1.1694	.9205	.3907	2.3559
23	.4014	.3907	.9205	.4245	68	1.1868	.9272	.3746	2.4751
24	.4189	.4067	.9135	.4452	69	1.2043	.9336	.3584	2.6051
25	.4363	.4226	.9063	.4663	**70**	1.2217	.9397	.3420	2.7475
26	.4538	.4384	.8988	.4877	71	1.2392	.9455	.3256	2.9042
27	.4712	.4540	.8910	.5095	72	1.2566	.9511	.3090	3.0777
28	.4887	.4695	.8829	.5317	73	1.2741	.9563	.2924	3.2709
29	.5061	.4848	.8746	.5543	74	1.2915	.9613	.2756	3.4874
30	.5236	.5000	.8660	.5774	**75**	1.3090	.9659	.2588	3.7321
31	.5411	.5150	.8572	.6009	76	1.3265	.9703	.2419	4.0108
32	.5585	.5299	.8480	.6249	77	1.3439	.9744	.2250	4.3315
33	.5760	.5446	.8387	.6494	78	1.3614	.9781	.2079	4.7046
34	.5934	.5592	.8290	.6745	79	1.3788	.9816	.1908	5.1446
35	.6109	.5736	.8192	.7002	**80**	1.3963	.9848	.1736	5.6713
36	.6283	.5878	.8090	.7265	81	1.4137	.9877	.1564	6.3138
37	.6458	.6018	.7986	.7536	82	1.4312	.9903	.1392	7.1154
38	.6632	.6157	.7880	.7813	83	1.4486	.9925	.1219	8.1443
39	.6807	.6293	.7771	.8098	84	1.4661	.9945	.1045	9.5144
40	.6981	.6428	.7660	.8391	**85**	1.4835	.9962	.0872	11.4301
41	.7156	.6561	.7547	.8693	86	1.5010	.9976	.0698	14.3007
42	.7330	.6691	.7431	.9004	87	1.5184	.9986	.0523	19.0811
43	.7505	.6820	.7314	.9325	88	1.5359	.9994	.0349	28.6363
44	.7679	.6947	.7193	.9657	89	1.5533	.9998	.0175	57.2900
45	.7854	.7071	.7071	1.0000	**90**	1.5708	1.0000	.0000	∞

ANSWERS TO EXERCISES

Pages 6, 7

1. $x = \frac{1}{3}, -2.$
2. $x = 1, -5.$
3. $x = \dfrac{-5 \pm \sqrt{13}}{6}.$
4. $x = \dfrac{-1 \pm \sqrt{-3}}{2}.$
5. $x = \pm 1, \pm \sqrt{2}.$
6. $x = \pm 1.$
7. $x = \pm \frac{1}{3}\sqrt{3}.$
8. $x = \dfrac{3 \pm \sqrt{13}}{2}.$
9. $x = \dfrac{-1 \pm \sqrt{17}}{4}.$
10. $x = \dfrac{1 \pm \sqrt{-3}}{2}.$
11. $x = \pm 1, 2.$
12. $x = 1, \dfrac{-1 \pm \sqrt{-3}}{2}.$
13. $x = \pm 1, \pm \sqrt{-1}.$
14. $x = \dfrac{\pm\sqrt{2} \pm \sqrt{-2}}{2}.$
15. $x = \dfrac{\pm\sqrt{3} \pm \sqrt{-5}}{2}.$

Page 9

1. $x = 1, -1, 2.$
2. $x = 2, 2, -\dfrac{5}{3}.$
3. $x = -3, \frac{5}{2}, \frac{5}{2}.$
4. $x = \frac{1}{2}, \frac{3}{2}, \frac{3}{2}.$
5. $x = 1, \frac{1}{2}(3 \pm \sqrt{29}).$
6. $x = -3, \frac{1}{2}(-1 \pm \sqrt{-3}).$
7. $x = -1, -1, \pm\frac{1}{2}.$
8. $x = -2, 3, \frac{3}{2}, -\frac{2}{3}.$
9. $x = 2, \frac{5}{3}, 1 \pm \sqrt{-2}.$
10. $x = 4, -\frac{5}{2}, \frac{1}{2}(3 \pm \sqrt{5}).$

Page 10

1. $x = -0.53, 0.65, 2.88.$
2. $x = 1.41, -0.7 \pm 2.1\sqrt{-1}.$
3. $x = 0.54, -0.8 \pm 1.1\sqrt{-1}.$
4. $x = 1.2, 2.9.$
5. $x = 0.7, -1.2.$

Page 12

1. $x > 1$ or $x < -2.$
2. $x > 1$ or $-1 < x < 0.$
3. $x < 1.$
4. $x < -1.53,$ or $-0.35 < x < 1.88.$

5. $x > 2$, $0 < x < \frac{2}{3}\sqrt{3}$, or $-2 < x < -\frac{2}{3}\sqrt{3}$.

7. $|x| < \frac{2}{3}\sqrt{3}$.

8. $|x| > \sqrt{2}$.

Page 14

1. $x = 1$, $y = 2$.
2. $x = 2$, $y = -1$, $z = 3$.
3. $x = 1$, $y = -1$, $z = 1$.
4. $x = 1$, $y = \frac{1}{2}$, $z = \frac{1}{3}$.
5. $x = \frac{2}{3}$, $y = -2$, $z = \frac{2}{5}$.
6. $x = -1$, $y = 1$, and $x = -\frac{8}{5}$, $y = -\frac{4}{5}$.
7. $h = 1$, $k = 2$, $r = \pm 5$, and $h = \frac{9}{49}$, $k = -\frac{26}{7}$, $r = \pm\frac{205}{49}$.
8. $x = 1, -1, 2, -2$. $y = -1, 1, -\frac{1}{2}, \frac{1}{2}$.
9. $x = y = z = \pm\sqrt{2}$.
10. $x = 1$, $y = -1$, $z = 2$, and $x = \frac{17}{13}$, $y = -\frac{14}{13}$, $z = \frac{23}{13}$.

Page 16

11. $x : y : z = 3 : 1 : 2$.

12. $x : y : z = 2 : -3 : 4$.

13. $x : y : z = 2 : -1 : 1$, or $-1 : 2 : 1$.

14. $x : y : z = 1 : \pm 1 : 1$, or $-2 : \pm\sqrt{-2} : 1$.

Page 29

1. $(\frac{5}{3}, 0)$.
2. $(3, 8)$.
3. $(5, 11)$, $(\frac{7}{3}, 3)$.
4. $AB : CD = 3 : -2$.
5. $(5, 3)$, $(-1, -5)$.

Page 31

1. $5 + 3\sqrt{5}$.

7. $(2 \pm 2\sqrt{3}, 5)$.

8. $(-\frac{1}{3}, 0)$.

9. $(\frac{17}{13}, 3\frac{17}{13})$.

10. $(\frac{9}{10}, \frac{1}{10})$.

11. $(2\frac{1}{5}, 1\frac{2}{5})$.

Page 34

7. $(-2, 2)$.

8. $[-\frac{1}{2}, -\frac{7}{2}]$, $[4, 1]$, $[-\frac{7}{2}, \frac{5}{2}]$.

9. $(-8, 5)$.

10. $(5, -1)$.

12. $24\frac{1}{2}$.

Page 38

2. $(0, \frac{1}{3})$, $(-4, 7)$.
3. $P\,(14, -13)$, $Q\,(6, -5)$.
4. $(-10, 31)$.
5. $D\,(15, -3)$.

8. $D\,(\frac{8}{3}, \frac{11}{3})$.
9. $P\,(1, 2\frac{2}{3})$.

12. $(2\frac{1}{5}, 4\frac{4}{5})$.

Pages 42, 43

1. $\frac{1}{2}$, -7, -2.
2. $\tan^{-1}(-0.8) = 141° \, 20'$.
9. $46° \, 51'$, $97° \, 8'$, $36° \, 2'$.
11. $7 \pm 5\sqrt{2}$.
12. $71° \, 34'$.
13. $\frac{5}{11}\sqrt{3} + \frac{8}{11}$.
15. $(4.933,\ 4.966)$,
 $(-6.933,\ -0.966)$.
16. $(-2\frac{6}{7},\ -3\frac{4}{7})$.

Pages 48, 49

2. $x - 3y = 5$.
4. $y = 3x + 9$.
5. $x + 5y = 2$.
6. $y^2 = 8y - 6x - 25$.
7. $3x - 4y + 6 = 0$.
9. $x^2 + y^2 = x + y + 14$.
11. $x^2 + y^2 - 4x - 6y = 12$.

Page 56

1. $y + 3 = \sqrt{3}\,(x + 1)$.
2. $y = 3x - 7$.
3. $x = 2$.
4. $y = 2$.
5. $2x + 8y = 17$.
6. $y - 5 = \pm\sqrt{3}\,(x - 3)$.
7. $6y - 9x + 2 = 0$.
8. $x - y + 2 = 0$.
9. $y = 5x - 3$, length $= 2\sqrt{26}$.
10. $x - 2y + 3 = 0$, $x - 2y + 8 = 0$,
 $2x + y - 4 = 0$, $2x + y - 9 = 0$.
11. $21x + 16y = 6$, $11x + 24y = 9$.

Pages 59, 60

14. $45°$, $56° \, 19'$, $78° \, 4'$.
15. $3x - 2y = 6$.
16. $y = 5x - 4$.
17. $y = x - 4$.
18. $8x + 9y = 7$.
19. $(1\frac{8}{17},\ -\frac{15}{17})$.
20. $5x + 3y = 18$.
21. 5, $\sqrt{10}$, $\sqrt{17}$.
22. $x - 5y + 8 = 0$.
23. $12x - 15y = 8$.
24. $4x - y + 9 = 0$.
25. A straight line perpendicular to the line through their centers.

Pages 63, 64

8. $x + y > 0$, $2x - 3y - 1 < 0$, $y - 2 < 0$.
9. $y + 3x - 4 > 0$, $3x - 2y - 1 > 0$, $y - 6x + 14 > 0$.
10. $\frac{1}{17}\sqrt{17}$.
11. $\frac{1}{13}\sqrt{13}$.
12. $\frac{2}{5}$.
13. $x\,(2\sqrt{2} - 1) - y\,(\sqrt{2} + 3) = \sqrt{2} - 2$.
14. $(\frac{16}{11},\ \frac{2}{11})$.
15. $(y - 2x)^2 = 14x + 18y - 56$.

Pages 68, 69.

1. $(0, 0)$, $r = 5$.
2. $(2, 0)$, $r = 2$.
3. $(0, \frac{3}{2})$, $r = \frac{1}{2}\sqrt{7}$.
4. $(-\frac{2}{3}, 0)$, $r = \frac{1}{3}\sqrt{7}$.
5. $(\frac{1}{2}, \frac{1}{2})$, $r = \frac{1}{2}\sqrt{2}$.
6. (a, a), $r = a\sqrt{2}$.
7. $(1, -2)$, $r = \sqrt{3}$.
8. The locus is the point $(1, 0)$.
11. $x^2 + y^2 - 5x + 4y - 46 = 0$.
12. $x^2 + y^2 - 2x - 2y = 11$.
13. $x^2 + y^2 - 2x - 2y + 1 = 0$, $x^2 + y^2 - 10x - 10y + 25 = 0$.
14. $x^2 + y^2 = 9 \pm 2\sqrt{8}$.
15. $(0, 2)$, $(\frac{6}{5}, -\frac{2}{5})$.
16. $x^2 + y^2 - 3x - 3y = 0$.
17. $(x - 6)^2 + (y - 2)^2 = 25$, $(x + 1)^2 + (y + 5)^2 = 25$.
18. $2x^2 + 2y^2 + 6x + 3y = 10$.
19. $x^2 + y^2 - 16x - 12y = 0$, $x^2 + y^2 + 4y = 0$.
20. $4x^2 + 4y^2 + x - y = 3$.
21. $(x - 5)^2 + (y - 14)^2 = 4$.
22. $x^2 + y^2 + 26x + 16y = 32$.
23. $x^2 + y^2 - (2\sqrt{10} - 6)y = 2\sqrt{10} - 6$.
24. $x = 2y$.

Pages 73, 74.

1. $(0, 0)$, $a = \sqrt{6}$, $b = \sqrt{3}$.
2. $(1, -2)$, $a = \frac{1}{2}$, $b = 1$.
3. $(\frac{1}{2}, \frac{1}{4})$, $a = \frac{1}{4}\sqrt{6}$, $b = \frac{1}{4}\sqrt{3}$.
4. $(1, -2)$, $a = 2$, $b = \sqrt{6}$.
5. $(-3, 1)$, $a = \sqrt{13}$, $b = \frac{1}{3}\sqrt{39}$.
6. The locus is the point $(1, -1)$.
7. $9x^2 + 4y^2 - 18x + 24y + 9 = 0$.
8. $4x^2 + y^2 + 16x - 8y + 16 = 0$.
9. $(x + y - 2)^2 + 16(x - y + 2)^2 = 32$.

Page 77.

1. Axis $y = 0$, vertex $(\frac{1}{2}, 0)$.
2. $y = 0$, $(\frac{1}{2}, 0)$.
3. $x = 1$, $(1, 2)$.
4. $x = -\frac{1}{2}$, $(-\frac{1}{2}, -\frac{9}{4})$.
5. $y = \frac{3}{2}$, $(-\frac{9}{4}, \frac{3}{2})$.
6. $x = \frac{3}{2}$, $(1\frac{1}{2}, 3\frac{1}{3})$.
7. $3y^2 = 16x$.
8. $5x^2 + 20x + 9y + 2 = 0$.
9. $\frac{29}{2}\sqrt{2}$.
10. $35\frac{5}{9}$ ft.

Page 83.

1. Center $(0, 2)$, $a = \sqrt{3}$, $b = 2$. Asymptotes, $y - 2 = \pm\frac{2}{3}\sqrt{3}\,x$.
2. Center $(0, -1)$, $a = \frac{1}{5}\sqrt{15}$, $b = \sqrt{3}$. Asymptotes, $y + 1 = \pm x\sqrt{5}$.
3. Center $(1, -1)$, $a = b = 2$. Asymptotes, $x = 1$ and $y = -1$. Axes, $x + y = 0$ and $x - y - 2 = 0$.
4. Center $(-1, 2)$, $a = \sqrt{3}$, $b = \sqrt{2}$. Asymptotes, $y - 2 = \pm\frac{2}{3}\sqrt{3}\,(x + 1)$.
5. The locus is two lines $x - y = 4$ and $x + y + 2 = 0$.
6. Center $(0, \frac{3}{2})$, $a = b = 2$. Asymptotes, $x = 0$ and $y = \frac{3}{2}$. Axes, $y - \frac{3}{2} = \pm x$.
7. $16(x - 2)^2 - (y + 1)^2 = \pm 16$.

8. $24(x+2)^2 - 5(y-1)^2 = 91.$
9. $4(3x+2y)^2 - 25(2x-3y)^2 = \pm 1300.$

Page 89.

1. The circle circumscribed about the square.
2. Two parabolas having the fixed diameter as common chord and with vertices at the middle points of the perpendicular radii.
3. A rectangular hyperbola passing through A and B.
4. A circle with center at the center of the triangle.
5. The circle passing through the vertices of the base angles, and tangent to the equal sides of the triangle.
6. A hyperbola.
9. A rectangular hyperbola.
10. Two circles passing through A and B with centers at the ends of the diameter perpendicular to AB.

Page 120.

1. $x = -2.$
2. $y = 3.$
3. $x - y = \sqrt{2}.$
4. $y = x\sqrt{3}.$
5. $x^2 + y^2 = 3x.$
6. $x^2 + y^2 = 4y.$
7. $x^2 + y^2 = \sqrt{2}(y - x).$
8. $x^2 + y^2 = x - y.$
9. $x^2 + y^2 = 4.$
10. $x^2 + y^2 - 2x - 2\sqrt{3}y + 3 = 0.$
11. $x^2 + y^2 - 2x - 2y + 1 = 0.$
12. $3x^2 + 4y^2 - 4x = 4.$
13. $4y^2 - 5x^2 - 36x = 36.$
14. $y^2 = 6x + 9.$
15. $x^2 = 4y + 4.$
16. $xy = 4x + 4y - 8.$
17. $xy = 2y - 3x.$
18. $x^2 - y^2 = y.$
19. $r(2\cos\theta - \sin\theta) = 1.$
20. $r = 4\cot\theta\csc\theta.$
21. $r = 2\cos\theta.$
22. $r^2 = 14\csc 2\theta.$
23. $r^2 = \sec 2\theta.$
24. $r = 4\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right).$
25. $r^2 + 2ar(\pm\cos\theta \pm \sin\theta) + a^2 = 0.$
26. $r\left[1 - \cos\left(\theta - \frac{\pi}{6}\right)\right] = 4.$
27. $r(3 - 2\sin\theta) = 3 - \sqrt{3}.$
28. $\sqrt{2}.$

Pages 126, 127.

35. $\theta = -\frac{\pi}{24},\ r = 3.285.$
36. $\left(a\sqrt{2}, \frac{\pi}{4}\right).$
37. $(0, 0),\ \left(\pm a, \frac{\pi}{2}\right),\ \left(\pm 2^{-\frac{1}{4}}a, \frac{\pi}{4}\right),\ \left(\pm 2^{-\frac{1}{4}}a, \frac{3}{4}\pi\right).$
38. $(0, 0),\ (.785\,a, \pm 25^\circ\,52'),\ (.409\,a, \pm 102^\circ\,4'),\ (.898\,a, \pm 148^\circ\,3').$
39. $\left(a, \pm\frac{\pi}{6}\right),\ \left(a, \pm\frac{5}{6}\pi\right).$

Pages 128, 129.

1. $r = a\cos\theta$.
2. $r = a(\sec\theta + \tan\theta)$.
3. $r(r\cos\theta - a) = k$. O is the origin and LK is perpendicular to OX at $(a, 0)$.
4. $r = a\sin 2\theta$. The length of the segment is $2a$.
5. $r = a + b\sec\theta$. The radius of the circle is $2a$ and the distance from the center to the fixed line is $2b$.
6. $r = 2a\tan\theta\sin\theta$. OA is the initial line and a the radius of circle.
7. $r = a(1 + \cos\theta)$, a cardioid.
8. $r = a(\csc\theta - 1)$.
9. $r\sin(\frac{1}{2}\theta) = a$.
10. $r = a\sec\theta + b$. The distance from O to BC is a and the constant distance is b.
11. $r = 2a\cos\theta + b$. The diameter through O is the initial line, the radius of the circle is a, and the constant distance is b.
12. $r = a\cos^2\theta$, a being the length of OA.
13. $r = a(1 - \tan^4\theta)\cos\theta$.
14. $r = \dfrac{c\sin\left(\dfrac{a}{b}\theta\right)}{\sin\left(\dfrac{c}{b}\theta\right)}$. The radii are a and b and the distance between centers is c. The origin is at the center of circle of radius a.

Pages 131, 132.

2. $x = a(1 + \tan\phi)$, $y = a\tan\phi$, $x - y = a$.
3. $x = a(1 + 2\sin^2\phi)$, $y = 2a\tan\phi\sin^2\phi$, $y^2(3a - x) = (x - a)^3$.
4. $x = 1 + \frac{1}{2}\cot\phi$, $y = \frac{1}{2} + \tan\phi$. $2xy = x + 2y$.
5. $r = a\sqrt{1 + \phi^2}$, $\theta = \phi - \tan^{-1}\phi$. $\theta = \dfrac{1}{a}\sqrt{r^2 - a^2} - \cos^{-1}\left(\dfrac{a}{r}\right)$.

Pages 137, 138.

3. $x^2 - y^2 = 4$.
4. $(y - x)^3 = 2(x + y)^2$.
5. $xy + x = 3y - 1$.
6. $4x^2 - 4\sqrt{3}xy + 4y^2 = 1$.
7. $x^2 - y^2 = 1$.
8. $\sqrt{x^2 + y^2} = \tan^{-1}\left(\dfrac{x}{y}\right)$.
9. $r = 1 - 2\theta^2$.
12. $x = a\cos^{-1}\left(\dfrac{a - y}{a}\right) - \sqrt{2ay - y^2}$.
13. $\theta^2(1 + r^2)^3 = r^2(3 - r^2)^2$.
14. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = (4a)^{\frac{2}{3}}$.
16. $r = \theta = t$.
17. $x = t^2\cos(1 + t)$, $y = t^2\sin(1 + t)$.
21. $(4, 4)$.

22. $(3, 4), (-4, -3)$.
23. $(\pm 0.5404\,a, 0.8414\,a)$.
24. $(\frac{4}{3}a, \pm\frac{4}{3}a\sqrt{2})$.
25. $x = a(1 + \cos 2\theta), \quad y = a \sin 2\theta$.
26. $x = \frac{4}{m^2}, \; y = \frac{4}{m}$.
27. $x = a\cos\phi, \; y = b\sin\phi$.
28. $x = a\sec\phi, \; y = b\tan\phi$.
29. $x = a\sin^3\phi, \; y = a\cos^3\phi$.
30. $x = m^2, \; y = m^3 - m - 2$.

Pages 140, 141.

1. $x = b\tan\phi \mp a\sin\phi, \quad y = \pm a\cos\phi$, the fixed point on the y-axis being $(0, b)$.
2. $x = k(1 + \cos^2\phi), \quad y = k(\tan\phi + \sin\phi\cos\phi)$.
3. $x = 2a\cot\phi, \quad y = 2a\sin^2\phi. \quad (x^2 + 4a^2)y = 8a^3$.
4. $x = \frac{a}{2}(1 + \cos\phi), \quad y = \frac{a}{2}(\sin\phi + \tan\frac{1}{2}\phi)$.
   $4xy^2 = (a - x)(a + 2x)^2$.
5. $x = (a - c\tan\phi)\sin^2\phi, \quad y = (a - c\tan\phi)\sin\phi\cos\phi$.
   $x(ay - cx) = y(x^2 + y^2). \quad r = (a - c\cot\theta)\cos\theta$.
6. $x = a\tan\phi, \quad y = a\cos 2\phi. \quad y(a^2 + x^2) = a^3 - ax^2$.
7. $x = a(\tan\phi + \sin\phi\cos\phi), \quad y = a(1 + \cos^2\phi)$.
   $y(x^2 + y^2) = a(x^2 + 2y^2). \quad r = a(\csc\theta + \sin\theta)$.
8. $x = 2a\cos^2\phi, y = 2a\sec\phi. \quad xy^2 = 8a^3, \; x \geqq 2a$.
   O is the origin, OA the x-axis, and $\phi = AOC$.
9. $r = a\sec^3\theta$.
10. $(x^2 + y^2 - 2a^2)^2 = a^3(5a \pm 4y)$. The fixed diameter is x-axis and the center of circle is origin.
11. $x = c\cos 2\phi - a\sin\phi + b\cos\phi, \quad y = c\sin 2\phi + a\cos\phi + b\sin\phi$. The radii are a and b and the distance between centers is $2c$. The x-axis passes through the centers and the origin is midway between them.
12. A rectangular hyperbola.
13. $x = -a\sin(\phi + B), \; y = b\sin(\phi - A)$. The curve is an ellipse.
14. $r = a\csc\phi, \quad \theta = \csc\phi + \cot\phi + \phi - \frac{\pi}{2} - 1$.
    The origin is at the center of circle, the initial line passes through the intersection of curve and circle, and ϕ is the angle formed at the pencil by the string.
15. $x = a\phi - b\sin\phi, \quad y = a - b\cos\phi$.
16. $x = \frac{a}{4}(3\cos\phi + \cos 3\phi), \quad y = \frac{a}{4}(3\sin\phi - \sin 3\phi). \quad x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.
    The radius of the fixed circle is a.
17. $y + 1 = 0$.

Pages 147–149.

3. $x^2 + 4y^2 = 4$.
4. $y^3 + 3x^2 + 16 = 0$.
5. $r = 2$.
6. $r = 4p \cot\theta \csc\theta$.
7. $2x - 3y = 0$.
8. $\dfrac{(x-a)^2}{a^2} + \dfrac{y^2}{b^2} = 1$.
9. $(x^2 + 4a^2)y + 2ax^2 = 0$.
10. $xy^2 = (x-a)^2(2a - x)$.
11. $x = a(\phi' + \sin\phi')$, $y = a(\cos\phi' - 1)$, where $\phi' = \phi - \pi$.
12. $x = \dfrac{a}{2}(2\cos\theta + \cos 2\theta)$, $y = \dfrac{a}{2}(2\sin\theta + \sin 2\theta)$.
13. $x + y = 0$, $2x - 3y = 0$.
14. $x^2 + y^2 = 11$.
20. $3x^2 + y^2 = 2$.
21. $r^2 \cos 2\theta = 2$.
22. $x^2 - y^2 = 8$.
23. $\dfrac{x^2}{4(\sqrt{2}-1)} - \dfrac{y^2}{4(\sqrt{2}+1)} = 1$.

Page 154.

4. Distance from the x-axis $\sqrt{y^2 + z^2}$, distance from the origin $\sqrt{x^2 + y^2 + z^2}$.
5. In the xy-plane $(1, 2, 0)$, on the x-axis $(1, 0, 0)$.
6. The projections on the y-axis are $1, -2, 1$.
7. $76° 22'$, $76° 22'$, $19° 28'$.
8. $5, -4$, and 3.
9. $(\frac{5}{2}, -\frac{1}{2}, \frac{3}{2})$.

Page 156.

3. $(1, 2, 3)$.
4. $(\frac{11}{3}, 3, -2)$, $(\frac{3}{2}, -\frac{1}{4}, \frac{5}{4})$.
5. The projection on the xy-plane is $[2, 2, 0]$.
7. $(5, 2, 5)$.

Pages 159, 160.

3. $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$.
4. $45°$ and $135°$.
5. $\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}, 0$.
6. $54° 44'$.
7. $\frac{1}{9}, \frac{4}{9}, \frac{8}{9}$.
8. $70° 32'$, $48° 12'$, $48° 12'$.
9. $71° 34'$, $71° 34'$, $36° 52'$.
11. $56° 1'$.

Pages 166, 167.

1. $\cos\alpha = \dfrac{1}{\pm\sqrt{21}}$, $\cos\beta = \dfrac{2}{\pm\sqrt{21}}$, $\cos\gamma = \dfrac{4}{\pm\sqrt{21}}$.

   The positive square roots correspond to one direction along the normal, the negative to the other. If a particular direction is desired, the proper sign is easily determined. In the following answers only one set of cosines is given.

2. $\cos\alpha = \dfrac{1}{\sqrt{11}}$, $\cos\beta = -\dfrac{1}{\sqrt{11}}$,
$\cos\gamma = \dfrac{3}{\sqrt{11}}$.

3. $\cos\alpha = \cos\beta = \cos\gamma = \frac{1}{3}\sqrt{3}$.

4. $\cos\alpha = \dfrac{2}{\sqrt{13}}$, $\cos\beta = -\dfrac{3}{\sqrt{13}}$,
$\cos\gamma = 0$.

5. $\cos\alpha = \frac{3}{5}$, $\cos\beta = 0$, $\cos\gamma = \frac{4}{5}$.

6. $\cos\alpha = \cos\beta = 0$, $\cos\gamma = 1$.

7. $x + y\sqrt{2} + z = 0$.

8. $3x - 5y + 4z + 2 = 0$.

9. $\dfrac{x}{1} + \dfrac{y}{2} + \dfrac{z}{3} = 1$.

12. $70° 32'$.

13. $29° 40'$.

Page 170.

17. $x^2 + y^2 + z^2 = a^2$, $r^2 + z^2 = a^2$,
$\rho = a$.

18. $x^2 + y^2 + z^2 = 2az$,
$r^2 + z^2 = 2az$, $\rho = 2a\cos\phi$.

19. $x^2 + y^2 = a^2$, $r = a$, $\rho = a\csc\phi$.

20. $x^2 + y^2 = z^2$, $r = z$, $\phi = \dfrac{\pi}{4}$.

21. $y^2 + z^2 = 2az$.

22. $x^2 = az$.

23. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

24. $\dfrac{x^2}{a^2} + \dfrac{y^2 + z^2}{b^2} = 1$.

25. $\dfrac{x^2}{a^2} - \dfrac{y^2 + z^2}{b^2} = 1$, or

$$\frac{x^2 + z^2}{a^2} - \frac{y^2}{b^2} = 1.$$

26. $y^2 + z^2 = ax$.

27. $(\sqrt{x^2 + z^2} - b)^2 + y^2 = a^2$.
A circle with center on the x-axis is rotated about the y-axis.

Pages 177, 178.

1. There are two sets of direction cosines differing in algebraic sign. One set is:

$$\cos\alpha = -\tfrac{2}{3},\ \cos\beta = \tfrac{2}{3},\ \cos\gamma = \tfrac{1}{3}.$$

2. $\cos\alpha = \dfrac{1}{\sqrt{6}}$, $\cos\beta = -\dfrac{1}{\sqrt{6}}$,
$\cos\gamma = \dfrac{2}{\sqrt{6}}$.

3. $\cos\alpha = -\dfrac{7}{\sqrt{78}}$, $\cos\beta = \dfrac{2}{\sqrt{78}}$,
$\cos\gamma = \dfrac{5}{\sqrt{78}}$.

4. $\cos\alpha = \cos\beta = 0$, $\cos\gamma = 1$.

5. $\dfrac{x-2}{1} = \dfrac{y-3}{1} = \dfrac{z+1}{3}$.

6. $\dfrac{x}{3} = \dfrac{y-1}{1} = \dfrac{z-2}{5}$.

7. $\dfrac{x-1}{1} = \dfrac{y-1}{2} = \dfrac{z-1}{-1}$.

8. $60°$.

9. $58° 31'$.

Page 180.

1. The projection on the xy-plane is

$$x^2 + y^2 + xy - x - y = 0,\ z = 0.$$

2. The projection on the yz-plane is

$$z = \pm a,\ x = 0.$$

3. The projection on the xz-plane is

$$z^2 = 2x^2 - 2x + 1, \quad y = 0.$$

11. $(-1, 1, 2)$, $(-\frac{1}{3}, \frac{1}{3}, \frac{4}{3})$.

Pages 181, 182.

1. The projection on the xy-plane is $x + y = 3$, $z = 0$.
2. The projection on the yz-plane is $y = \sin(\frac{1}{2}z)$, $x = 0$.
3. The projection on the xz-plane is $x = z \sin z$, $y = 0$.
5. $x = a\theta \cos\theta$, $y = a\theta \sin\theta$, $z = k\theta$.
6. $z = k\theta$.
7. $y = x\sqrt{2}$, $z = x - .000064\,x^2$, $z = .7071\,y - .000032\,y^2$.

INDEX

The numbers refer to the pages